Infrared-Finite Amplitudes for Massless Gauge Theories

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Abstract

We present a method to construct infrared-finite amplitudes for gauge theories with massless fermions. Rather than computing S-matrix elements between usual states of the Fock space we construct order-by-order in perturbation theory dressed states that incorporate all long-range interactions. The S-matrix elements between these states are shown to be free from soft and collinear singularities. As an explicit example we consider the process $e^+e^- \to 2$ jets at next-to-leading order in the strong coupling. We verify by explicit calculation that the amplitudes are infrared finite and recover the well-known result for the total cross section $e^+e^- \to$ hadrons.

1 Introduction

Soft and collinear singularities which arise in gauge theories such as QED and QCD are usually dealt with by summing up physically indistinguishable cross sections [1, 2]. This results in a cancellation of these singularities for so-called infrared-safe observables and therefore reliable theoretical predictions can be made for such observables. Even though this is a perfectly valid approach to the problem it is useful to investigate the origin of these singularities and to explore the possibility of avoiding them altogether.

The origin of the problem lies in the long-range nature of the interactions. As a consequence the usual in and out states do not evolve in time asymptotically according to the free Hamiltonian. It is this breakdown of the standard assumption that results in the non-existence of the scattering operator. Thus, if we want to avoid infrared singularities from the outset we have to construct true asymptotic states and compute transition amplitudes between them [3, 4].

This program has been carried out initially for QED with massive fermions [5] and then many steps have been made to extend it to soft singularities in non-abelian theories [6, 7, 8]. It is possible to construct asymptotic states (generalized coherent states) which include multiple soft gluon emission to all orders in the coupling [9]. It can be shown that the S-matrix between such states is free of soft singularities [10, 11]. Apart from the more complicated structure of the soft singularities due to the self-interaction of the gauge bosons there is the additional complication of collinear singularities in a non-abelian gauge theory. Due to the collinear singularities the asymptotic Hamiltonian is more complicated [12, 13, 14, 15] and the prospect of being able to include these effects to all orders in perturbation theory are not very promising. But the idea of constructing an asymptotic Hamiltonian that takes into account the asymptotic dynamics and using the corresponding evolution operator to dress the usual states [16, 17] can still be applied and is not tied to any particular theory. In particular, four-point interactions that are present in non-abelian gauge theories can be incorporated [18].

In the present work we investigate the practical feasibility of constructing asymptotic states whose S-matrix elements are free from soft and collinear singularities. We are not so much interested in general considerations but rather try to establish a method to define and explicitly compute infrared

1

finite amplitudes order-by-order in perturbation theory. Apart from the conceptional advantage of avoiding divergent amplitudes such a method would have a variety of practical advantages. Obviously, the finiteness of the amplitudes would facilitate a completely numerical approach to the calculation of amplitudes. This also applies to the combination of fixed order results with parton shower Monte Carlo programs.

After some general remarks about infrared singularities in Section 2 we discuss how to construct the dressed states in Section 3. This discussion will be completely general. In order to test the practicability of our method we consider a simple process in Section 4. We compute the infrared finite amplitudes for $e^+e^- \to 2$ jets at next-to-leading order in the strong coupling and verify that upon integration over the phase space they reproduce the well-known total cross section for $e^+e^- \to$ hadrons. Technical details of this section are relegated to an Appendix. Finally, in Section 5, we present a summary and suggestions on how to improve the method to make applications to more complicated processes more feasible.

2   Infrared singularities

We are concerned with gauge theories with massless fermions. Such theories are plagued by infrared singularities. Infrared singularities are related to either arbitrarily soft gauge bosons or arbitrarily collinear gauge bosons and/or fermions.[1] More precisely, if we define our external states in the usual way by acting with creation operators on the vacuum, then higher-order S-matrix elements between such external states contain infrared divergences.

Many attempts have been made to define amplitudes that are well defined, i.e. do not contain such singularities. However, most attempts were restricted to soft singularities. In particular it has been shown that in an abelian gauge theory with massive fermions it is possible to define external states whose S-matrix elements are free from infrared singularities to all orders in perturbation theory [5].

An abelian gauge theory with massive fermions does not contain collinear singularities. This simplifies the situation considerably. As soon as we consider the non-abelian case, however, we cannot avoid the appearance of

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[1] We use the term "infrared singularities" for both, soft and collinear singularities

collinear divergences. The reason is that in the non-abelian case a massless gauge boson can split into two arbitrarily collinear gauge bosons. Such a splitting results in a collinear singularity. Thus, giving the fermions a mass does not protect us from collinear singularities.

The aim of this work is to investigate the possibility of defining infrared-finite amplitudes for a massless, non-abelian gauge theory. The application we have in mind is, of course, QCD. In most applications of perturbative QCD the quarks (at least the light flavors) are treated as massless. Since we will have to deal with collinear singularities anyway in a non-abelian theory, we might as well take the common approach and treat the quarks as massless.

2.1 The conventional approach

Before we tackle the problem of defining infrared-finite amplitudes in a situation where collinear singularities have to be taken into account as well, it might be useful to remind ourselves how infrared singularities are dealt with in the conventional approach.

In the conventional approach, which in the following we shall call the cross-section approach, we evaluate amplitudes between conventional external states that are obtained simply by acting with creation operators on the vacuum. Since such amplitudes contain infrared singularities we regularize them. In virtually all applications dimensional regularization is used, whereby the amplitudes are evaluated in $D \equiv 4-2\epsilon$ dimensions. The infrared singularities then reveal themselves as poles in $1/\epsilon$.

In order to relate these amplitudes to measurable quantities, we will have to integrate the (squared) amplitudes over the phase space, weighted by a measurement function that defines the quantity we are interested in. The key point is that we can only obtain theoretical predictions for quantities that are infrared safe. An infrared safe quantity is one that does not depend on whether or not a parton emits an arbitrarily soft gluon. Also it must not depend on whether or not a parton splits into two collinear partons. For such quantities the infrared singularities present in the amplitudes are cancelled by infrared singularities that appear due to the phase space integration and after the cancellation the regulator can be removed. Even though there are various general methods available to perform such calculations at next-to-leading order [19] the appearance of infrared singularities make the phase-

space integration a non-trivial problem.

2.2 Asymptotic interactions

One of the basic assumptions in the derivation of Greens functions or transition amplitudes is that in the limit $t \to \pm\infty$ the external states are free. Thus, it is assumed that all interactions vanish rapidly enough so that they can be neglected in the distant past and future. More precisely, we assume that for any exact state vector $|\Psi(t)\rangle$ (in the Schrödinger picture) of the full Hamiltonian H we can find a corresponding state vector $|\Phi(t)\rangle$ of the free Hamiltonian H_0 such that

$$|\Psi(t)\rangle = e^{-itH}|\Psi(0)\rangle \;\; \to \;\; |\Phi(t)\rangle = e^{-itH_0}|\Phi(0)\rangle \tag{1}$$

for $t \to \pm\infty$. We then specify the in- and out-states $|\Phi_i\rangle$ and $\langle\Phi_f|$ by a complete set of quantum numbers of H_0 and compute the transition amplitudes

$$\langle\Phi_f|\Omega_-^\dagger\Omega_+|\Phi_i\rangle \equiv \langle\Phi_f|S|\Phi_i\rangle \tag{2}$$

where we expressed the scattering operator S in terms of the usual Møller operators

$$\Omega_\mp \equiv \lim_{\tau \to \pm\infty} e^{i\tau H}e^{-i\tau H_0} \tag{3}$$

Since S commutes with H_0 energy is conserved and the amplitude, eq. (2), is proportional to $\delta(E_i - E_f)$. This is a point we will come back to below.

The important point is that in a gauge theory with conventional external states the above mentioned assumption is simply incorrect. The interactions due to massless gauge bosons are long-range interactions and do *not* vanish rapidly enough for $t \to \pm\infty$. It is this violation of our basic assumption that results in ill-defined, infrared-singular amplitudes. In other words, given an exact state $|\Psi(t)\rangle$, in general there does not exist a free state $|\Phi(t)\rangle$ such that eq. (1) is satisfied.

If we want to tackle the problem at its root, we will have to compute amplitudes between modified external states. The long-range interactions that cause the problem will have to be included in the external states themselves. This amounts to replacing our basic assumption with the following: we can find an asymptotic Hamiltonian H_A such that for any exact state

4

vector $|\Psi(t)\rangle$ we can find an asymptotic state $|\Xi(t)\rangle$ such that

$$|\Psi(t)\rangle = e^{-it\,H}|\Psi(0)\rangle \;\rightarrow\; |\Xi(t)\rangle = e^{-it\,H_A}|\Xi(0)\rangle \tag{4}$$

for $t \rightarrow \pm\infty$. We then define modified Møller operators

$$\Omega_{A\mp} \equiv \lim_{\tau\rightarrow\pm\infty} e^{i\tau H}e^{-i\tau H_A} \tag{5}$$

and compute modified S-matrix elements

$$\langle\Xi_f|\Omega^\dagger_{A-}\Omega_{A+}|\Xi_i\rangle \equiv \langle\Xi_f|S_A|\Xi_i\rangle \tag{6}$$

Once all true long-range interactions are included in the definition of the asymptotic Hamiltonian we are guaranteed that S-matrix elements between such asymptotic states are free from infrared singularities [10, 11]. These S-matrix elements in turn are related to measurable quantities. In the same way as several (squared) amplitudes contribute to a physical cross section in the conventional approach, there will be several (squared) S-matrix elements between asymptotic states contributing to an observable. The crucial difference is that contrary to the cross-section method all these contributions are separately finite.

To find the asymptotic Hamiltonian we have to split the interaction Hamiltonian into a "hard" and a "soft" piece

$$H_I = H_H(\Delta) + H_S(\Delta) \tag{7}$$

and define $H_A(\Delta) \equiv H_0 + H_S(\Delta)$. The separation of the Hamiltonian into two pieces is by no means unique. The only requirement is that $H_S(\Delta)$ includes all true long-range interactions. Thus, the emission of a soft gauge boson and the splitting of a parton into two collinear partons has to be included. But there is a lot of freedom on how precisely we make the split between soft and hard interactions. To indicate this arbitrariness we include the parameter Δ in the notation. Later, we will often use the abbreviated notation $H_\Delta \equiv H_S(\Delta)$. Let us stress that even though we will call $H_\Delta \equiv H_S(\Delta)$ the soft Hamiltonian, it includes all long-range interactions that potentially give rise to infrared singularities. In particular, the soft Hamiltonian also includes the splitting of a parton into two collinear partons.

In what follows we will define H_A and the asymptotic states more carefully. We then show how infrared-finite S-matrix elements are related to conventional amplitudes. Finally, we consider a simple explicit example, $e^+e^- \rightarrow 2$ jets at next-to-leading order.

3 Infrared-finite amplitudes

In order to obtain infrared-finite amplitudes we have to find true asymptotic states and evaluate the (modified) S-matrix elements between such states, as given in eq. (6). Measurable cross sections then are constructed out of these infrared-finite matrix elements.

Since S_A commutes with H_A we conclude from eq. (6) that energy is also conserved for the modified transition amplitudes. However, given an asymptotic Hamiltonian it is generally not possible to find the corresponding eigenstates $|\Xi_i\rangle$. First of all, being an eigenstate of the asymptotic Hamiltonian which includes all soft interactions, the true states $|\Xi_i\rangle$ would correspond to quasi hadrons. In our strict perturbative approach we will never be able to describe bound states and, at each order in perturbation theory, the states $|\Xi_i\rangle$ will correspond to some sort of "jet-like" states. In particular, these states are colored and will have to be related to hadronic states using a hadronization model. This is as in the cross-section approach and is an issue that we do not address in this paper. Still, we have to relate the asymptotic states $|\Xi_i\rangle$ to conventional free states $|\Phi_i\rangle$ order by order in perturbation theory. We are then led to compute matrix elements

$$\mathcal{M}_{fi} \equiv \langle \Phi_f | S_A | \Phi_i \rangle \tag{8}$$

order by order in perturbation theory and relate them to physical (infrared safe) cross sections. It has been argued previously that matrix elements as given in eq. (8) are also free of soft singularities [10, 11]. In the present article we extend this to include collinear singularities (see also [14]). For more details on this issue we refer to Section 3.5.

3.1 Notation and conventions

Before we proceed let us fix our notation and conventions. Whereas part of the discussion so far was done in the Schrödinger picture we will now turn to the interaction picture. Thus all operators and states are now to be understood to be given in the interaction picture.

To start with we construct the usual states of the Fock space

$$|q_i(p_i) \dots \bar{q}_j(p_j) \dots g_k(p_k) \dots\rangle \equiv \prod_i b^\dagger(p_i) \prod_j d^\dagger(p_j) \prod_k a^\dagger(p_k) |0\rangle \tag{9}$$

where $b^\dagger$, $d^\dagger$ and $a^\dagger$ denote the creation operators for fermions, antifermions and gauge bosons respectively and we suppressed the helicity labels. We will generically denote such states by $|i\rangle$ and $\langle f|$. Of course, we have to keep in mind that the states as given in eq. (9) are not normalizable and we tacitly assume they have been smeared with test functions. Thus, we are really concerned with wave packets. However, we assume they are sharply peaked around a certain value of the momentum such as to represent a particle beam with (nearly) uniform, sharp momentum.

The creation and annihilation operators satisfy the usual (anti)commutation relations

$$
\begin{align}
[a(\lambda_1, \vec{k}_1), a^\dagger(\lambda_2, \vec{k}_2)] &= -(2\pi)^3 2\omega(\vec{k}_1) g_{\lambda_1 \lambda_2} \, \delta(\vec{k}_1 - \vec{k}_2) \tag{10} \\
\{b(r_1, \vec{k}_1), b^\dagger(r_2, \vec{k}_2)\} &= (2\pi)^3 2\omega(\vec{k}_1) \delta_{r_1 r_2} \, \delta(\vec{k}_1 - \vec{k}_2) \tag{11} \\
\{d(r_1, \vec{k}_1), d^\dagger(r_2, \vec{k}_2)\} &= (2\pi)^3 2\omega(\vec{k}_1) \delta_{r_1 r_2} \, \delta(\vec{k}_1 - \vec{k}_2) \tag{12}
\end{align}
$$

with $\omega(\vec{k}_i) \equiv |\vec{k}_i|$. Note that the ordering used in eq. (9) implies a certain phase convention. Of course, all amplitudes are only defined up to such a convention.

The field operators are given by

$$
\begin{align}
\Psi_\alpha &= \int d\widetilde{k} \left(u_\alpha(r, \vec{k}) \, b(r, \vec{k}) \, e^{-ikx} + v_\alpha(r, \vec{k}) \, d^\dagger(r, \vec{k}) \, e^{+ikx} \right) \tag{13} \\
\overline{\Psi}_\alpha &= \int d\widetilde{k} \left(\bar{u}_\alpha(r, \vec{k}) \, b^\dagger(r, \vec{k}) \, e^{+ikx} + \bar{v}_\alpha(r, \vec{k}) \, d(r, \vec{k}) \, e^{-ikx} \right) \tag{14} \\
A_\mu &= \int d\widetilde{k} \left(\varepsilon_\mu(\lambda, \vec{k}) \, a(\lambda, \vec{k}) \, e^{-ikx} + \varepsilon_\mu^*(\lambda, \vec{k}) \, a^\dagger(\lambda, \vec{k}) \, e^{+ikx} \right) \tag{15}
\end{align}
$$

where we defined

$$
d\widetilde{k} \equiv \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1} 2\omega(\vec{k})} \sum_{1,2} \tag{16}
$$

and the sum is over the two helicities of the fermions or gauge bosons respectively.

Once the interaction Hamiltonian is given we can compute the evolution operator and obtain in the interaction picture

$$
U(t, t_0) \equiv T \, \exp\left(-i \int_{t_0}^t dt \, H_I(t) \right) \tag{17}
$$

7

The Møller operators are given by $\Omega_\pm = U(0, \mp\infty)$ and, thus, the scattering operator S is related to the evolution operator

$$S = \Omega_-^\dagger \Omega_+ = U(+\infty, 0)U(0, -\infty) \tag{18}$$

This allows us to find the S-matrix elements between some initial and final state

$$\langle f|S|i\rangle = \langle f|T \exp\left(-i \int_{-\infty}^{+\infty} dt\, H_I(t)\right) |i\rangle \tag{19}$$

where $|i\rangle$ and $\langle f|$ are states as defined in eq. (9). Inserting the explicit form of H_I into eq. (19) allows us to compute S-matrix elements. Of course, in practice such a calculation is nothing but the computation of the corresponding Feynman diagrams.

3.2   Definition of infrared-finite amplitudes

In analogy to eq. (17) we define a soft evolution operator

$$U_\Delta(t, t_0) \equiv T \exp\left(-i \int_{t_0}^{t} dt\, H_\Delta(t)\right) \tag{20}$$

where we only include the soft Hamiltonian $H_\Delta(t) \equiv H_S(\Delta, t)$. Acting on a certain state, the soft evolution operator modifies this state by allowing for soft and collinear emissions. Then, the usual Feynman-Dyson scattering matrix S can be decomposed as

$$S = U(+\infty, 0)U(0, -\infty) \equiv \Omega_{\Delta-}^\dagger S_A(\Delta)\Omega_{\Delta+} \tag{21}$$

where we have introduced the soft Møller operators $\Omega_{\Delta\pm} \equiv U_\Delta(0, \mp\infty)$. More explicitly, we have

$$\begin{aligned}
\Omega_{\Delta-}^\dagger &\equiv T \exp\left(-i \int_0^\infty dt H_\Delta(t)\right) \\
&= 1 - i \int_0^\infty dt\, H_\Delta(t) + (-i)^2 \int_0^\infty dt \int_0^t dt'\, H_\Delta(t)H_\Delta(t') + \ldots \\
&= 1 - i \int_0^\infty dt\, H_\Delta(t) + \frac{(-i)^2}{2!} \int_0^\infty dt \int_0^\infty dt'\, T\{H_\Delta(t)H_\Delta(t')\} + \ldots
\end{aligned} \tag{22}$$

Eq. (21) defines a modified scattering operator $S_A(\Delta)$. This operator has the crucial property that it includes at least one hard interaction and, therefore,

8

matrix elements $\langle f|S_A(\Delta)|i\rangle$ of this operator with ordinary external initial and final states as defined in eq. (9) have no infrared singularities. If we define dressed initial and final states, $|\{i\}\rangle$ and $\langle\{f\}|$ according to

$$|\{i\}\rangle \equiv \Omega_{\Delta+}^{\dagger}|i\rangle \tag{23}$$

$$\langle\{f\}| \equiv \langle f|\Omega_{\Delta-} \tag{24}$$

then

$$\langle\{f\}|S|\{i\}\rangle = \langle f|S_A(\Delta)|i\rangle \tag{25}$$

Thus, the S-matrix elements of dressed states are free of infrared singularities. We should stress that dressed states are not asymptotic states, i.e. they are not eigenstates of the asymptotic Hamiltonian.

Let us look at a dressed final state somewhat more carefully. We obtain a dressed final state by acting with $\Omega_{\Delta-}$ on a final state as defined in eq. (9). We denote this dressed state by adding curly brackets.

$$_f\langle\{q(p_i)\ldots\bar{q}(p_j)\ldots g(p_k)\ldots\}| = \langle q(p_i)\ldots\bar{q}(p_j)\ldots g(p_k)\ldots|\Omega_{\Delta-} \tag{26}$$

Once the asymptotic Hamiltonian is fixed eq. (26) is a unique relation, order by order in perturbation theory, between an ordinary final state $\langle f|$ and the corresponding dressed final state $\langle\{f\}|$. A similar relation holds for dressed initial states.

$$|\{q(p_i)\ldots\bar{q}(p_j)\ldots g(p_k)\ldots\}\rangle_i = \Omega_{\Delta+}^{\dagger}|q(p_i)\ldots\bar{q}(p_j)\ldots g(p_k)\ldots\rangle \tag{27}$$

In what follows we will suppress the labels f and i but keep in mind that the states $|\{q(p_i)\ldots\bar{q}(p_j)\ldots g(p_k)\ldots\}\rangle$ and $\langle\{q(p_i)\ldots\bar{q}(p_j)\ldots g(p_k)\ldots\}|$ are not conjugates of each other. Also, we would like to stress that all these states are states in the usual Fock space. Of course, this implicitly assumes that we use some kind of regularization for the infrared singularities in intermediate steps.

The soft Møller operators dress the usual non-interacting external states with a cloud of soft and collinear partons. Since the infrared behavior of H_Δ and the full interaction Hamiltonian are the same by construction, this dressing generates infrared singularities that cancel those generated by the full scattering operator.

There are two main differences between the soft(/collinear) Møller operator, eq. (22), and the usual scattering operator, eq. (19). Firstly, $\Omega_{\Delta\pm}$

involve only the soft part H_Δ of the interaction Hamiltonian. Secondly, the time integration in the soft Møller operator runs only from 0 to ∞ rather than from $-\infty$ to ∞.

The fact that the time integration is restricted to $t > 0$ is related to the loss of Lorentz invariance in the amplitudes $\mathcal{M}_{fi}$, eq. (8). This is to be expected since S_A does not commute with H_0 and, therefore, $\mathcal{M}_{fi}$ is generally not proportional to an energy conserving $\delta(E_i - E_f)$. Instead, individual parts of the amplitude will have δ-functions with different energy arguments (see eq. (36)). The difference between these arguments determines the amount by which energy conservation can be violated in $\mathcal{M}_{fi}$ and is related to the parameter Δ. In the limit $\Delta \to 0$ the amount by which energy can be violated tends to 0. Thus, the parameter Δ determines how much the initial wave packets are distorted through the evolution with the soft Møller operators. We will come back to these issues in Section 3.7.

3.3   Factorization of modified S-matrix elements

We now turn to the question on how to compute the infrared-finite amplitudes defined in eq. (25) and how they are related to ordinary amplitudes.

A possible approach is to start from the right hand side of eq. (25). This would involve using the explicit form of S_A, given below in eq. (36) to compute the amplitudes. As we argue in Section 3.5 the structure of S_A is such that no infrared singularities occur. This opens up the possibility of evaluating the amplitudes numerically. We have to keep in mind, however, that there are still ultraviolet singularities which will have to be removed by renormalization. In order to take an entirely numerical approach the renormalization procedure would have to be done at the integrand level [20].

We will take a somewhat different approach in that we start from the left hand side of eq. (25). We relate the infrared finite amplitude to ordinary amplitudes by inserting a complete set of states twice

$$\langle \{f\}|S|\{i\}\rangle = \langle f|\Omega_{\Delta-} \, S \, \Omega_{\Delta+}^\dagger|i\rangle = \langle f|\Omega_{\Delta-}|f'\rangle \otimes \langle f'|S|i'\rangle \otimes \langle i'|\Omega_{\Delta+}^\dagger|i\rangle \quad (28)$$

Note that in the final expression all states are ordinary Fock space states as defined in eq. (9). In this way, the infrared finite amplitude is split into three pieces. First, there is an ordinary S-matrix element, $\langle f'|S|i'\rangle$. The other two factors are dressing factors for the initial and final state. All these pieces are

10

infrared divergent and only the complete amplitude is infrared finite, order by order in perturbation theory. The ultraviolet singularities appear only in $\langle f'|S|i'\rangle$ and are dealt with as usual by renormalization. The symbol $\otimes$ denotes integration over all momenta and summation over all helicities of the state under consideration. Thus, for say $\langle f'| = \langle q(p_1, r_1)\bar{q}(p_2, r_2)g(p_3, r_3)| \equiv \langle q_{p1}\bar{q}_{p2}g_{p3}|$ we have

$$|f'\rangle \otimes \langle f'| = \sum_{r_1\,r_2\,r_3} \int d\tilde{p}_1 d\tilde{p}_2 d\tilde{p}_3 \; |q_{p1}\bar{q}_{p2}g_{p3}\rangle \langle q_{p1}\bar{q}_{p2}g_{p3}| \tag{29}$$

We should stress that eq. (28) implies that the dressing is not done for each external parton separately. The dressing factors $\langle f|\Omega_{\Delta-}|f'\rangle$ do contain terms that factorize into separate contributions for each parton, but they also contain color correlated contributions.

3.4   Dressing factors

As we have seen in eq. (28) infrared-finite amplitudes are composed of three factors. First, there is an ordinary amplitude, $\langle f'|S|i'\rangle$, computed in the usual way using ordinary Feynman rules. Then there are the two dressing factors, one for the initial and one for the final state. The calculation of these dressing factors is somewhat different from the calculation of ordinary amplitudes and it is useful to look at this in some more detail.

For concreteness we consider the calculation of a final state dressing factor. The starting point is eq. (22). Let us stress again that since the time integration in eq. (28) is from 0 to ∞ we break Lorentz invariance right from the beginning. Of course, in the final result for a physical quantity Lorentz invariance will be restored. In fact, the calculation has many features of (old-fashioned) time-ordered perturbation theory. Most notably, all particles will be on-shell. Three-momentum will be conserved in all vertices, but energy will not be conserved.

A typical term of the (asymptotic) Hamiltonian that gives rise to an n-point interaction has the form

$$\int d\vec{x} \int \prod_{i=1}^{n} d\tilde{k}_i \; V(\vec{k}_i)\Theta(\Delta) \, e^{i\vec{x}\cdot\sum_i \sigma_i \vec{k}_i} e^{-it\sum \sigma_i \omega(\vec{k}_i)} \tag{30}$$

where $\omega(\vec{k}_j) \equiv |\vec{k}_j|$ denotes the energy of the particles and the sign σ_i is positive (negative) for incoming (outgoing) particles. $V(\vec{k}_i)$ is made up of creation

and annihilation operators, eventually accompanied by spinors and/or polarization vectors and a certain power of the coupling constant. The range of integration over the momenta is restricted to the singular regions. This is indicated in the notation by $\Theta(\Delta)$. The precise form of this function is not important at the moment. After performing the $\mathrm{d}\vec{x}$ integration we obtain the momentum conserving delta function $(2\pi)^{D-1}\delta^{(D-1)}(\sum \sigma_i \vec{k}_i)$. However, since the t integration is restricted to $t \geq 0$ we do not obtain an energy conserving δ function. Rather we have to introduce the usual adiabatic factor $0^+ > 0$ and use

$$\int_0^\infty \mathrm{d}t\, e^{-i\omega t} \to \int_0^\infty \mathrm{d}t\, e^{-i\omega t} e^{-t0^+} = \frac{-i}{\omega - i0^+} \tag{31}$$

Of course, if the t integration was restricted to $t \leq 0$ we would have

$$\int_{-\infty}^0 \mathrm{d}t\, e^{-i\omega t} \to \int_{-\infty}^0 \mathrm{d}t\, e^{-i\omega t} e^{+t0^+} = \frac{i}{\omega + i0^+} \tag{32}$$

and the sum of eq. (31) and eq. (32) indeed results in $2\pi\delta(\omega)$.

To summarize, for an n-point vertex in the calculation of a dressing factor for a final state we have to use

$$\int \prod_{i=1}^n \mathrm{d}\widetilde{k}_i\, (2\pi)^{D-1}\delta^{(D-1)}\left(\sum \sigma_i \vec{k}_i\right) \frac{\Theta(\Delta)}{\sum \sigma_i \omega(\vec{k}_i) - i0^+}\, V(\vec{k}_i) \tag{33}$$

Were it not for the $\Theta(\Delta)$ function and the restriction of the t-integration to $t \geq 0$ this would lead to the standard Feynman rule.

3.5 Finiteness of modified S-matrix elements

In this subsection we substantiate our claim that matrix elements as defined in eq. (8) or eq. (25) are free from collinear and soft singularities. We start from the definition

$$S_A(\Delta) = \Omega_{\Delta-} S\, \Omega_{\Delta+}^\dagger \tag{34}$$

and use the explicit form of the soft Møller operator and S to express $S_A(\Delta)$ in terms of H_Δ and H_H. Furthermore, we observe that according to eq. (30) the time dependence of the Hamiltonian $H(t_j)$ is given by

$$H_H(t_j) = h_j\, e^{-it_j\, \varpi_j}; \quad H_\Delta(t_j) = s_j\, e^{-it_j\, \varpi_j} \tag{35}$$

12

where $\varpi_j \equiv \sum \sigma_i \omega(\vec{k_i})$ is the sum of the energies of the particles associated with the corresponding n-point vertex and h_j and s_j are time independent. Performing the algebra and the t-integrations we obtain up to third order

$$
\begin{aligned}
S_A \;=\; & 1 - 2i\pi\, h_1\, \delta(\varpi_1) && (36)\\
& +\; 2i\pi\, h_1 h_2\, \frac{\delta(\varpi_1 + \varpi_2)}{\varpi_1} + 2i\pi\, [h_1, s_2]\, \frac{\delta(\varpi_1) - \delta(\varpi_1 + \varpi_2)}{\varpi_2}\\
& +\; 2i\pi\, h_1(h_2 + s_2)h_3\, \frac{\delta(\varpi_1 + \varpi_2 + \varpi_3)}{\varpi_1 \varpi_3}\\
& -\; 2i\pi\, [[h_1, s_2], s_3]\, \left(\frac{\delta(\varpi_1 + \varpi_2) - \delta(\varpi_1 + \varpi_2 + \varpi_3)}{\varpi_1 \varpi_3} + \frac{\delta(\varpi_1)}{\varpi_2(\varpi_2 + \varpi_3)} \right)\\
& -\; 2i\pi\, s_1 h_2 h_3\, \frac{\delta(\varpi_2 + \varpi_3) - \delta(\varpi_1 + \varpi_2 + \varpi_3)}{\varpi_1 \varpi_3}\\
& -\; 2i\pi\, h_1 h_2 s_3\, \frac{\delta(\varpi_1 + \varpi_2) - \delta(\varpi_1 + \varpi_2 + \varpi_3)}{\varpi_1 \varpi_3}
\end{aligned}
$$

First of all we notice that all the purely soft terms $s_1 s_2 \ldots$ vanish. This holds to all orders and is crucial to ensure that S_A is free from infrared singularities. Infrared singularities potentially arise if $\varpi_i \to 0$. This corresponds to either a soft or collinear emission at the corresponding vertex. Let us now go through the terms in eq. (36) and check that for none of them such a singularity can occur. For this to be true we have to define h_i such that it vanishes for $\varpi_i \to 0$. This can be achieved by choosing the $\Theta(\Delta)$ in eq. (30) accordingly.

We start by looking at the second order terms, given in the second line of eq. (36). The only potential singularity in the first term is $\varpi_1 \to 0$. This is harmless since $h_1 = 0$ in this limit. In the second term we have the potential singularity $\varpi_2 \to 0$ which is not prevented by s_2. However, in this limit the term is proportional to $\delta(\varpi_1)$ and the same argument as for the first order term applies.

The arguments for the third order terms, given in the third to sixth line of eq. (36) are similar. The only dangerous limits in the third line term for example are $\varpi_1 \to 0$ and $\varpi_3 \to 0$. Both of these are prevented by the presence of h_1 and h_3. Considering the term in the fourth line, we first note that $\delta(\varpi_1)h_1 = 0$. As a result there is no problem with the limit $\varpi_2 \to 0$ and $\varpi_2 \to -\varpi_3$. Furthermore, the singularity in the limit $\varpi_1 \to 0$ is prevented by the presence of h_1 and the limit $\varpi_3 \to 0$ is made harmless by the combination of δ functions. Similarly, the terms in the fifth and sixth line are finite in the

limit $\varpi_1 \to 0$ and $\varpi_3 \to 0$. Thus we see that (up to this order) there are no singularities in S_A as long as h_i is chosen to vanish for $\varpi \to 0$.

We mention again that S_A does not only contain terms proportional to $\delta(\varpi_1 + \varpi_2 + \varpi_3)$ but also terms with "incomplete" δ-functions. These are the energy violating terms mentioned above. We also remark that the absence of terms containing $1/(\varpi_i + \varpi_j)$ in S_A (the corresponding term in the fourth line of eq. (36) vanishes) justifies our initial claim that all infrared singularities are related to limits $\varpi_i \to 0$.

3.6 Construction of infrared finite amplitudes

The expression given in eq. (28) is a (double) sum over all possible intermediate states $|f'\rangle\langle f'|$ and $|i'\rangle\langle i'|$. However, if we compute the amplitude to a certain order in the coupling constant, only a very limited number of intermediate states contribute. It is for example clear that at order $\mathcal{O}(g^0)$ the dressing factor $\langle f|\Omega_{\Delta-}|f'\rangle$ is zero, unless $f = f'$. From this we see that at leading order in perturbation theory the amplitude $\langle \{f\}|S|\{i\}\rangle$ is the same as $\langle f|S|i\rangle$.

Including higher-order corrections this identity will, of course, not hold any longer. At order $\mathcal{O}(g^1)$ the states f and f' can be different. To get a non-vanishing contribution they must be related either by adding a (soft or collinear) gluon or by exchanging a quark-antiquark pair by a gluon.

In order to illustrate this in more detail, let us consider a concrete process. To simplify matters we consider a case with no partons in the initial state. What we have in mind is for example the process $e^+e^- \to \gamma \to$ jets. As long as we treat this process at leading order in the electromagnetic coupling but at higher order in the strong coupling, g, we encounter only final state singularities. Thus, for the purpose of understanding how the dressing removes the infrared singularities we can restrict ourselves to the final state partons and treat the initial state simply as $|0\rangle$.

Before writing down eq. (28) more explicitly for the process under consideration, let us introduce a somewhat more compact notation. We will denote the momenta and helicities of the partons in the intermediate state f' by q_i and s_i respectively and use the notation $q_{qi} \equiv q(\vec{q}_i, s_i)$ etc. The momenta and helicities of the partons in the final state f on the other hand will be

14

denoted by p_i and r_i and we use $q_{pi} \equiv q(\vec{p}_i, r_i)$. The order $\mathcal{O}(g^n)$ terms of the dressing factors are then denoted by

$$g^n \, \mathcal{W}^{(n)}(q_{p1}, \bar{q}_{p2}, g_{p3} \ldots; q_{q1}, \bar{q}_{q2} \ldots) \equiv \tag{37}$$
$$\langle q(\vec{p}_1, r_1) \bar{q}(\vec{p}_2, r_2) g(\vec{p}_3, r_3) \ldots | \Omega_{\Delta -} | q(\vec{q}_1, s_1) \bar{q}(\vec{q}_2, s_2) \ldots \rangle \Big|_{g^n}$$

Similarly, we denote the order $\mathcal{O}(g^n)$ terms of the amplitude by

$$g^n \, \mathcal{A}^{(n)}(q(\vec{q}_1, s_1), \bar{q}(\vec{q}_2, s_2), g(\vec{q}_3, s_3) \ldots; \gamma) \equiv \tag{38}$$
$$\langle q(\vec{q}_1, s_1) \bar{q}(\vec{q}_2, s_2) g(\vec{q}_3, s_3) \ldots | S | 0 \rangle \Big|_{g^n} \equiv g^n \, \mathcal{A}^{(n)}(q_{q1}, \bar{q}_{q2}, g_{q3} \ldots; \gamma)$$

and we introduce a notation for the infrared finite amplitudes

$$g^n \, \mathcal{A}^{(n)}(\{q_1(\vec{p}_1, r_1), \bar{q}_2(\vec{p}_2, r_2), g_3(\vec{p}_3, r_3) \ldots\}; \gamma) \equiv \tag{39}$$
$$\langle \{q(\vec{p}_1, r_1) \bar{q}(\vec{p}_2, r_2) g(\vec{p}_3, r_3) \ldots\} | S | 0 \rangle \Big|_{g^n} \equiv g^n \, \mathcal{A}^{(n)}(\{q_{p1}, \bar{q}_{p2}, g_{p3} \ldots\}; \gamma)$$

We always make use of the convention that the helicity associated with momentum $\vec{p}_i$ is r_i whereas the helicity associated with momentum $\vec{q}_i$ is s_i.

Let us now use eq. (28) to write down the infrared finite amplitude $\langle \{q(p_1, r_1) \bar{q}(p_2, r_2)\} | S | 0 \rangle$ order by order in perturbation theory. At leading order we have

$$\begin{aligned}
\mathcal{A}^{(0)}(\{q_{p1}, \bar{q}_{p2}\}; \gamma) &\equiv \langle \{q(p_1, r_1) \bar{q}(p_2, r_2)\} | S | 0 \rangle \Big|_{g^0} \tag{40} \\
&= \mathcal{W}^{(0)}(q_{p1}, \bar{q}_{p2}; q_{q1}, \bar{q}_{q2}) \otimes \mathcal{A}^{(0)}(q_{q1}, \bar{q}_{q2}; \gamma) \\
&= \mathcal{A}^{(0)}(q_{p1}, \bar{q}_{p2}; \gamma)
\end{aligned}$$

where in the last step we used

$$\mathcal{W}^{(0)}(q_{p1}, \bar{q}_{p2}; q_{q1}, \bar{q}_{q2}) = \tag{41}$$
$$(2\pi)^3 2\omega(\vec{p}_1) \delta_{r_1 s_1} \delta(\vec{p}_1 - \vec{q}_1) \, (2\pi)^3 2\omega(\vec{p}_2) \delta_{r_2 s_2} \delta(\vec{p}_2 - \vec{q}_2)$$

Eq. (41) is simply obtained by noting that $\Omega_{\Delta -} = 1$ at $\mathcal{O}(g^0)$, eq. (22), and using the (anti)commutation relations eqs.(10), (11) and (12).

At $\mathcal{O}(g)$ the amplitude is zero because for every intermediate state f' either the dressing factor $\langle f | \Omega_{\Delta -} | f' \rangle$ or the amplitude $\langle f' | S | 0 \rangle$ vanishes.

At $\mathcal{O}(g^2)$ the situation is more interesting. We have

$$
\begin{aligned}
\mathcal{A}^{(2)}(\{q_{p1}, \bar{q}_{p2}\}; \gamma) & \equiv \langle\{q(p_1, r_1)\bar{q}(p_2, r_2)\}|S|0\rangle\big|_{g^2} \qquad\qquad (42) \\
& = \mathcal{W}^{(0)}(q_{p1}, \bar{q}_{p2}; q_{q1}, \bar{q}_{q2}) \otimes \mathcal{A}^{(2)}(q_{q1}, \bar{q}_{q2}; \gamma) \\
& + \mathcal{W}^{(2)}(q_{p1}, \bar{q}_{p2}; q_{q1}, \bar{q}_{q2}) \otimes \mathcal{A}^{(0)}(q_{q1}, \bar{q}_{q2}; \gamma) \\
& + \mathcal{W}^{(1)}(q_{p1}, \bar{q}_{p2}; q_{q1}, \bar{q}_{q2}, g_{q3}) \otimes \mathcal{A}^{(1)}(q_{q1}, \bar{q}_{q2}, g_{q3}; \gamma)
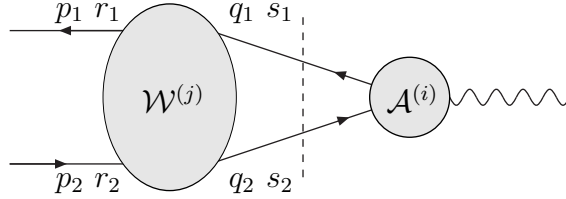\end{aligned}
$$

Figure 1: Cut diagrams for 2-particle intermediate state. The term $\mathcal{W}^{(0)} \otimes \mathcal{A}^{(2)}$ of eq. (43) corresponds to $j = 0, i = 2$ and $\mathcal{W}^{(2)} \otimes \mathcal{A}^{(0)}$ corresponds to $j = 2, i = 0$.

The first term on the right hand side of eq. (42) is nothing but the usual one-loop amplitude multiplied by the $\mathcal{O}(g^0)$ dressing factor and, using eq. (41), can be written as

$$
\mathcal{W}^{(0)}(q_{p1}, \bar{q}_{p2}; q_{q1}, \bar{q}_{q2}) \otimes \mathcal{A}^{(2)}(q_{q1}, \bar{q}_{q2}; \gamma) = \mathcal{A}^{(2)}(q_{p1}, \bar{q}_{p2}; \gamma) \qquad (43)
$$

The second term is also a two-particle cut term, but this time it is the usual tree-level amplitude multiplied by the next-to-leading order dressing factor. These two terms are shown in Figure 1.

The third term in eq. (42) is of a somewhat different nature as it is a three-particle cut diagram, as illustrated in Figure 2. The dressing factor $\mathcal{W}^{(1)}(q_{p1}, \bar{q}_{p2}; q_{q1}, \bar{q}_{q2}, g_{q3})$ is zero unless the gluon g_{q3} is either soft or collinear to the quark or antiquark. Thus, the dressing factor projects out the infrared singular piece of the bremsstrahlung amplitude. This is exactly the piece that is needed to render the full amplitude $\mathcal{A}^{(2)}(\{q_{p1}, \bar{q}_{p2}\}; \gamma)$ finite.

In the next section we will calculate this amplitude explicitly and check that the infrared singularities present in the three terms of eq. (42) cancel in the sum.
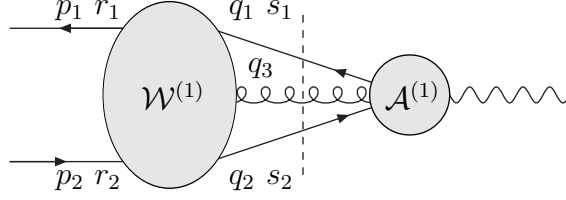
Figure 2: Cut diagram for 3-particle intermediate state.

The construction of the amplitude at higher orders in g follows the same pattern. For any odd power of g the amplitude vanishes for the same reason as it vanishes at $\mathcal{O}(g)$. At $\mathcal{O}(g^4)$ it is given by

$$
\begin{aligned}
\mathcal{A}^{(4)}(\{q_{p1}, \bar{q}_{p2}\}; \gamma) = & \\
& \mathcal{W}^{(0)}(q_{p1}, \bar{q}_{p2}; q_{q1}, \bar{q}_{q2}) \otimes \mathcal{A}^{(4)}(q_{q1}, \bar{q}_{q2}; \gamma) \\
+ & \mathcal{W}^{(2)}(q_{p1}, \bar{q}_{p2}; q_{q1}, \bar{q}_{q2}) \otimes \mathcal{A}^{(2)}(q_{q1}, \bar{q}_{q2}; \gamma) \\
+ & \mathcal{W}^{(4)}(q_{p1}, \bar{q}_{p2}; q_{q1}, \bar{q}_{q2}) \otimes \mathcal{A}^{(0)}(q_{q1}, \bar{q}_{q2}; \gamma) \\
+ & \mathcal{W}^{(1)}(q_{p1}, \bar{q}_{p2}; q_{q1}, \bar{q}_{q2}, g_{q3}) \otimes \mathcal{A}^{(3)}(q_{q1}, \bar{q}_{q2}, g_{q3}; \gamma) \\
+ & \mathcal{W}^{(3)}(q_{p1}, \bar{q}_{p2}; q_{q1}, \bar{q}_{q2}, g_{q3}) \otimes \mathcal{A}^{(1)}(q_{q1}, \bar{q}_{q2}, g_{q3}; \gamma) \\
+ & \mathcal{W}^{(2)}(q_{p1}, \bar{q}_{p2}; q_{q1}, \bar{q}_{q2}, g_{q3}, g_{q4}) \otimes \mathcal{A}^{(2)}(q_{q1}, \bar{q}_{q2}, g_{q3}, g_{q4}; \gamma) \\
+ & \mathcal{W}^{(2)}(q_{p1}, \bar{q}_{p2}; q_{q1}, \bar{q}_{q2}, q_{q3}, \bar{q}_{q4}) \otimes \mathcal{A}^{(2)}(q_{q1}, \bar{q}_{q2}, q_{q3}, \bar{q}_{q4}; \gamma)
\end{aligned}
\tag{44}
$$

The separate terms in eq. (44) are infrared divergent but in the sum all these divergences cancel. This can be seen by looking at a particular Feynman diagram, for example the one shown in Figure 3, and realizing that eq. (44) is nothing but the sum over all possible cuts. Since the dressing factors are constructed such that in the infrared limit they correspond to the usual amplitudes it is clear that the infrared singularities in $\mathcal{A}^{(4)}(\{q_{p1}, \bar{q}_{p2}\}; \gamma)$ have to cancel in the same way as they cancel in ordinary cut diagrams. The first term of eq. (44) corresponds to the ordinary two-loop amplitude and is represented by cut 1. The other two-particle cuts, the second and third term, are represented by cut 2 and 3. There are two three-particle cut terms, term 4 and 5. Finally, for the diagram under consideration, there is one four-particle cut contribution, namely term 6. For a certain Feynman diagram not all terms of eq. (44) are present. In our case, the last term of eq. (44)

17

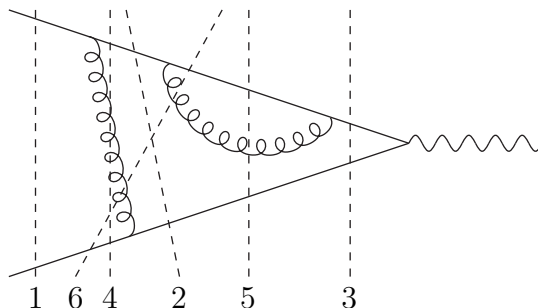which is another four-particle cut contribution is missing.

Figure 3: All possible cuts of a Feynman diagram representing the various terms of eq. (44).

We should stress that our approach to construct infrared finite amplitudes is by no means restricted to amplitudes with final state singularities only. Initial state singularities are dealt with by dressing the initial state, as can be seen in eq. (28).

In fact, the dressing of the initial state would even be needed for processes as discussed above, i.e. with say only a γ in the initial state. Above and in the rest of this paper we have excluded any QED vertices from the soft Hamiltonian even though there is a potential collinear singularity at this vertex. We do this because we treat the incoming photon as off-shell and so it will generate no infrared singularities.

If we were to include such vertices in the soft Hamiltonian then we would generate many more diagrams with non-vanishing initial-state dressing factors such as $\mathcal{W}(q(\vec{q}_1, s_1), \bar{q}(\vec{q}_2, s_2), g(\vec{q}_3, s_3); \gamma)$. We would find though, that all such extra contributes would cancel as all diagrams with purely soft vertices cancel as described in Section 3.5.

3.7 From amplitudes to cross sections

Once the infrared finite amplitudes have been computed, they can be used to compute cross sections for observables related to these amplitudes. The

18

procedure to obtain cross sections from amplitudes depends to some extent on the external states we use and deserves some further considerations.

In the cross-section approach we usually deal with amplitudes that are proportional to a four-dimensional delta function. Upon taking the absolute value squared, this leaves us with the problem of interpreting the square of a delta function. Usually this is dealt with in a rather non-rigorous manner by putting the system in a four dimensional box of size $V \cdot T$. The square of the delta function is then interpreted as $V \cdot T$ times a single delta function. The factor $V \cdot T$ is cancelled by taking into account the normalization of the states and the flux factor, which leaves us with a cross section proportional to a single four-dimensional delta function, expressing conservation of four momentum.

The appearance of the square of the delta function is of course related to the fact that we usually work with non-normalizable states with a sharp value of momentum and energy. In a more rigorous treatment within the cross-section approach the in and out states would have to be written as wave packets, sharply peaked around a certain value of momentum and energy. It can then be shown that the spreading of the wave packet during the scattering process can be safely neglected [21]. As mentioned in Section 2.1 the precise definition of the measurable quantity is given in terms of a measurement function. This is a function of the partonic momenta. If we are dealing with wave packets rather than sharp-momentum states, the measurement function has to be defined in terms of these wave packets. However, as long as we deal with wave packets whose spread is well below any experimental resolution, we can simply use the normal measurement function with the partonic momentum replaced by the central value of the wave packet and we get the same result as in the above mentioned, less rigorous approach [21].

Let us now turn to the situation we encounter if we work with infrared-finite amlitudes, defined in eq. (8). As mentioned before, the amplitude is then not proportional to an energy conserving delta function, even if we were to start with the usual non-normalizable states. Following the proper treatment with wave packets, we think of the states $|i\rangle$ and $\langle f|$ (or $|\Phi_i\rangle$ and $\langle \Phi_f|$) as sharply peaked wave packets. The states $|\{i\}\rangle$ and $\langle \{f\}|$ as defined in eqs. (23) and (24) are also wave packets. Through the action of the Møller operators, their spread is larger than the spread of $|i\rangle$ and $\langle f|$ and depends crucially on the parameter Δ. If we choose Δ small enough

such that the spread of the wave packets related to the states $|\{i\}\rangle$ and $\langle\{f\}|$ is still smaller than any experimental resolution, we can still compute any measurable cross section by using the standard measurement function with the partonic momenta replaced by the central value of the wave packet.

The important point is that we must be able to express any measurable quantity in terms of the states $|\{i\}\rangle$ and $\langle\{f\}|$. However, since the states $|\{i\}\rangle$ and $\langle\{f\}|$ differ from $|i\rangle$ and $\langle f|$ only by soft and collinear interactions, this is nothing but the requirement that the quantity we are dealing with is infrared safe. Indeed, the requirement of infrared safety states that the quantity must not depend on wheter or not a parton emits another arbitrarily soft or collinear parton. But in the limit $\Delta \to 0$ the soft Møller operators do precisely this. Thus, choosing Δ small enough ensures that the construction of the measurable quantity in terms of the partonic momenta is not affected by the soft Møller operators.

This solves the problem on how to obtain differential cross sections, once the infrared-finite amplitudes are known, in principle. In practice, the explicit implementation of this programme is far from trivial and requires further investigations. We mention for example that choosing Δ very small might result in numerical problems, similar to the so called binning problem in the standard approach. If, on the other hand, we choose Δ too large (relative to the experimental resoluion) the infrared-finite amplitudes are too inclusive to allow the computation of any possible physical quantity. It has been advocated before that the most convenient choice of H_A is the one that precisely corresponds to the experimental resolution [14]. While this might be true in principle, we think that this is not a practicable way to proceed, since then the asymptotic Hamiltonian would depend on the details of the experiment.

4   An example $e^+e^- \to$ 2 jets at NLO

We consider the process $e^+e^- \to \gamma(P) \to$ 2 jets. At leading order there is only one partonic process that contributes, $e^+e^- \to q\bar{q}$. However, at next-to-leading order there is also the process $e^+e^- \to q\bar{q}g$. Since the initial state does not interact strongly we can restrict our considerations to the process $\gamma^*(P) \to$ 2 jets.

4.1 The conventional approach

In the conventional cross-section approach we compute the amplitudes for the two partonic processes

$$\mathcal{A}(q_{p1}, \bar{q}_{p2}; \gamma(P)) = \\
\mathcal{A}^{(0)}(q_{p1}, \bar{q}_{p2}; \gamma(P)) + g^2 \mathcal{A}^{(2)}(q_{p1}, \bar{q}_{p2}; \gamma(P)) + \mathcal{O}(g^4) \tag{45}$$

and

$$\mathcal{A}(q_{p1}, \bar{q}_{p2}, g_{p3}; \gamma(P)) = g\mathcal{A}^{(1)}(q_{p1}, \bar{q}_{p2}, g_{p3}; \gamma(P)) + \mathcal{O}(g^3) \tag{46}$$

Upon squaring the amplitude and integration over the phase space we obtain

$$d\sigma = d\sigma_0 + g^2 d\sigma_{q\bar{q}} + g^2 d\sigma_{q\bar{q}g} + \mathcal{O}(g^4) \tag{47}$$

where

$$d\sigma_0 \sim |\mathcal{A}^{(0)}(q_{p1}, \bar{q}_{p2}; \gamma(P))|^2 \tag{48}$$

$$d\sigma_{q\bar{q}} \sim 2\mathrm{Re}\left[\mathcal{A}^{(0)}(q_{p1}, \bar{q}_{p2}; \gamma(P))\mathcal{A}^{(1)*}(q_{p1}, \bar{q}_{p2}; \gamma(P))\right] \tag{49}$$

$$d\sigma_{q\bar{q}g} \sim |\mathcal{A}^{(1)}(q_{p1}, \bar{q}_{p2}, g_{p3}; \gamma(P))|^2 \tag{50}$$

The virtual cross section, $d\sigma_{q\bar{q}}$ and the real cross section, $d\sigma_{q\bar{q}g}$ both contain infrared singularities and only when combined to form an infrared safe observable do these divergences cancel. For the total cross section for example we obtain

$$\sigma_0 = \frac{4\pi\alpha_{\mathrm{em}}^2 e_q^2 N_c}{3s} \tag{51}$$

$$\sigma_{q\bar{q}} = \sigma_0 \, C_F \frac{\alpha_s}{\pi} c_\Gamma \left(\frac{2}{\epsilon^2} + \frac{3}{2\epsilon} + \frac{19}{4} - \frac{\pi^2}{2}\right) \tag{52}$$

$$\sigma_{q\bar{q}g} = \sigma_0 \, C_F \frac{\alpha_s}{\pi} c_\Gamma \left(-\frac{2}{\epsilon^2} - \frac{3}{2\epsilon} - 4 + \frac{\pi^2}{2}\right) \tag{53}$$

where $c_\Gamma = 1 + \mathcal{O}(\epsilon)$ and $C_F = (N_c^2 - 1)/(2N_c) = 4/3$. Thus, at next to leading order the total cross section is given by

$$\sigma_1 = \sigma_{q\bar{q}} + \sigma_{q\bar{q}g} = \sigma_0 \left(1 + \frac{\alpha_s}{4\pi} 3C_F\right) \tag{54}$$

4.2 The infrared finite amplitudes

In terms of infrared finite amplitudes, at next-to-leading order a cross section is also made up of two contributions. The two amplitudes that contribute are those with the final states $\langle\{q_{p1}\bar{q}_{p2}\}|$ and $\langle\{q_{p1}\bar{q}_{p2}g_{p3}\}|$. However, the crucial point is that both these amplitudes are infrared finite. Up to the order in g required they are given by

$$\mathcal{A}(\{q_{p1},\bar{q}_{p2}\};\gamma) \equiv \langle\{q_{p1}\bar{q}_{p2}\}|S|0\rangle = \tag{55}$$
$$\int d\widetilde{q}_1 d\widetilde{q}_2 \; \mathcal{W}^{(0)}(q_{p1},\bar{q}_{p2};q_{q1},\bar{q}_{q2}) \times \mathcal{A}^{(0)}(q_{q1},\bar{q}_{q2};\gamma(P))$$
$$+ \int d\widetilde{q}_1 d\widetilde{q}_2 \; g^2 \, \mathcal{W}^{(0)}(q_{p1},\bar{q}_{p2};q_{q1},\bar{q}_{q2}) \times \mathcal{A}^{(2)}(q_{q1},\bar{q}_{q2};\gamma(P))$$
$$+ \int d\widetilde{q}_1 d\widetilde{q}_2 \; g^2 \, \mathcal{W}^{(2)}(q_{p1},\bar{q}_{p2};q_{q1},\bar{q}_{q2}) \times \mathcal{A}^{(0)}(q_{q1},\bar{q}_{q2};\gamma(P))$$
$$+ \int d\widetilde{q}_1 d\widetilde{q}_2 d\widetilde{q}_3 \; g^2 \, \mathcal{W}^{(1)}(q_{p1},\bar{q}_{p2};q_{q1},\bar{q}_{q2},g_{q3})$$
$$\times \mathcal{A}^{(1)}(q_{q1},\bar{q}_{q2},g_{q3};\gamma(P)) \; + \; \mathcal{O}(g^4)$$

and

$$\mathcal{A}(\{q_{p1},\bar{q}_{p2},g_{p3}\};\gamma) \equiv \langle\{q_{p1}\bar{q}_{p2}g_{p3}\}|S|0\rangle = \tag{56}$$
$$\int d\widetilde{q}_1 d\widetilde{q}_2 \; g \, \mathcal{W}^{(1)}(q_{p1},\bar{q}_{p2},g_{p3};q_{q1},\bar{q}_{q2}) \times \mathcal{A}^{(0)}(q_{q1},\bar{q}_{q2};\gamma(P))$$
$$+ \int d\widetilde{q}_1 d\widetilde{q}_2 d\widetilde{q}_3 \; g \, \mathcal{W}^{(0)}(q_{p1},\bar{q}_{p2},g_{p3};q_{q1},\bar{q}_{q2},g_{q3})$$
$$\times \mathcal{A}^{(1)}(q_{q1},\bar{q}_{q2},g_{q3};\gamma(P)) \; + \; \mathcal{O}(g^3)$$

where a sum over the spin/helicities of the intermediate particles is understood to be included in $\int d\widetilde{q}_i$, eq. (16).

4.3 The asymptotic Hamiltonian

Before we can proceed with the calculation of the infrared finite amplitudes we have to define the asymptotic Hamiltonian H_Δ. Once we have H_Δ we can obtain the Møller operator, eq. (22), and use it to construct the dressed states, eq. (26), order by order in perturbation theory.

The only condition on H_Δ is that it includes all long-range interactions from the original Hamiltonian. In order to separate these long range soft and

collinear emission terms from the hard emission terms we need to introduce (at least) one parameter which we denote by Δ. The dependence of the asymptotic Hamiltonian on this parameter is indicated in the notation H_Δ. Once these terms are incorporated into the asymptotic Hamiltonian we are free to include any other terms from the original Hamiltonian that we wish, as these will only produce finite Δ-dependent contributions to the two final amplitudes. It is clear from eq. (28) that for the final result this Δ dependence has to cancel.

In our case the only term of the interaction Hamiltonian we wish to include in H_Δ is the quark gluon interaction vertex,

$$H_I \equiv g \int d\vec{x} : \overline{\Psi} T^a \gamma^\mu \Psi : A^a_\mu \tag{57}$$

Using eqs. (13,14) and (15) we see that H_I consists of eight terms

$$
\begin{aligned}
H_I \;=\;& g\,T^a \int d\widetilde{k}_1 d\widetilde{k}_2 d\widetilde{k}_3 \sum_{i=1}^{8} V_i(\vec{k}_1, \vec{k}_2, \vec{k}_3) \\
& \times \exp\left(-it \sum_{j=1}^{3} \sigma_{ij}\omega(\vec{k}_j)\right) \delta^{(D-1)}\left(\sum_{j=1}^{3} \sigma_{ij}\vec{k}_j\right),
\end{aligned}
\tag{58}
$$

where (suppressing the helicity and color labels)

$$
\begin{aligned}
V_1 &= b^\dagger(\vec{k}_1)b(\vec{k}_2)a(\vec{k}_3) \cdot \bar{u}(\vec{k}_1)\slashed{\epsilon}(\vec{k}_3)u(\vec{k}_2), \\
V_2 &= b^\dagger(\vec{k}_1)d^\dagger(\vec{k}_2)a(\vec{k}_3) \cdot \bar{u}(\vec{k}_1)\slashed{\epsilon}(\vec{k}_3)v(\vec{k}_2), \\
V_3 &= d(\vec{k}_1)b(\vec{k}_2)a(\vec{k}_3) \cdot \bar{v}(\vec{k}_1)\slashed{\epsilon}(\vec{k}_3)u(\vec{k}_2), \\
V_4 &= -d^\dagger(\vec{k}_1)d(\vec{k}_2)a(\vec{k}_3) \cdot \bar{v}(\vec{k}_2)\slashed{\epsilon}(\vec{k}_3)v(\vec{k}_1), \\
V_5 &= b^\dagger(\vec{k}_1)b(\vec{k}_2)a^\dagger(\vec{k}_3) \cdot \bar{u}(\vec{k}_1)\slashed{\epsilon}^*(\vec{k}_3)u(\vec{k}_2), \\
V_6 &= d(\vec{k}_1)b(\vec{k}_2)a^\dagger(\vec{k}_3) \cdot \bar{v}(\vec{k}_1)\slashed{\epsilon}^*(\vec{k}_3)u(\vec{k}_2), \\
V_7 &= b^\dagger(\vec{k}_1)d^\dagger(\vec{k}_2)a^\dagger(\vec{k}_3) \cdot \bar{u}(\vec{k}_1)\slashed{\epsilon}^*(\vec{k}_3)v(\vec{k}_2), \\
V_8 &= -d^\dagger(\vec{k}_1)d(\vec{k}_2)a^\dagger(\vec{k}_3) \cdot \bar{v}(\vec{k}_2)\slashed{\epsilon}^*(\vec{k}_3)v(\vec{k}_1)
\end{aligned}
\tag{59}
$$

The sign factors σ_{ij} are $+1$ (-1) for incoming (outgoing) particles.

As we only have to include terms in H_Δ that contribute in the singular regions we are free to exclude the V_i in eq. (59) for which $\sum \sigma_i \omega(\vec{k}_i)$ can never equal zero with all $\omega(\vec{k}_i) \geq 0$ such that not all of the $\omega(\vec{k}_i) = 0$. We can see

from eq. (33) that such terms will always be finite. From the remaining terms we choose only those that give a singularity in the physically relevant soft or collinear regions. This means that for our example we can exclude V_2 and V_6, as these only go singular when the two incoming or outgoing quarks from the vertex are collinear. We emphasize that for more general processes these terms have to be included in the asymptotic Hamiltonian.

We can confine the remaining terms even further as we are free to choose the form of the finite part of H_Δ. We restrict the integration of the momenta $\vec{k}_1, \vec{k}_2$ and $\vec{k}_3$ to just the potentially singular regions. This restriction is achieved here by including a theta function, $\Theta(\Delta_i(\vec{k}_1, \vec{k}_2, \vec{k}_3))$ in each V_i from eq. (59) which will appear in H_Δ. The form of $\Delta_i(\vec{k}_1, \vec{k}_2, \vec{k}_3)$ is completely arbitrary as long as $\Theta \to 1$ in the soft and collinear limits.

The form of the Θ function that we will take for this example is,

$$\Theta(\Delta_i(\vec{k}_1, \vec{k}_2, \vec{k}_3)) \equiv \Theta(\Delta - |\sum_j \sigma_{ij}\omega(\vec{k}_j)|). \tag{60}$$

This choice of $\Delta(\vec{k}_1, \vec{k}_2, \vec{k}_3)$ is particularly appropriate because as we see in eq. (33), $\sum_j \sigma_{ij}\omega(\vec{k}_j)$ is the exact form that the singular terms take. This theta function therefore restricts the integral to just the regions close to these singular limits.

By splitting up the covariant vertex into pieces and restricting the integration to just the singular regions we are removing the manifest Lorentz and gauge invariance from the amplitudes. Physical observables will though be Lorentz and gauge invariant as we are effectively just performing a unitary transformation (as we have regulated the $\Omega_\pm$ operators) on a known Lorentz and gauge invariant result.

To summarize, for our asymptotic Hamiltonian we take just the vertices V_1, V_4, V_5 and V_8, giving,

$$
\begin{aligned}
H_\Delta &= g \int \mathrm{d}\widetilde{k}_1 \mathrm{d}\widetilde{k}_2 \mathrm{d}\widetilde{k}_3 \sum_{i=1,4,5,8} \left\{ V_i(\vec{k}_1, \vec{k}_2, \vec{k}_3) \exp\left(-it \sum_{j=1}^3 \sigma_{ij}\omega(\vec{k}_j)\right) \right. \\
&\quad \left. \delta^{(D-1)}\left(\sum_{j=1}^3 \sigma_{ij}\vec{k}_j\right) \Theta(\Delta - |\sum_j \sigma_{ij}\omega(\vec{k}_j)|) \right\}.
\end{aligned}
\tag{61}
$$

24

4.4 Diagrammatic rules for the asymptotic regions

We can now take eq. (61) and use it in eq. (22) to form the asymptotic operator. We could then go on to calculate the dressed states of eq. (37) with this operator defined between suitable in and out states by using the commutation relations between a, b, d, $a^\dagger$, $b^\dagger$, $d^\dagger$ and time-ordered perturbation theory. However it can be shown that there are a set of diagrammatic rules for the asymptotic region which behave in a similar way to Feynman diagrams in normal perturbative field theory. Using these we rules we can simplify the calculation.

These diagrammatic rules consist of vertex and propagator 'like' objects, but unlike normal Feynman diagrams we must take all time orderings of the vertices into account. This is because we base the evaluation of the amplitude in the asymptotic region on time ordered perturbation theory. As mentioned before energy is not conserved at each vertex and since the range of the time integration in the Møller operators is from 0 to ∞ there is no overall energy conservation.

As there is a time ordering to the vertices we have both absorption and emission rules. These are defined in Figure 4 with time flowing from right to left.

$$
\equiv (ig)T^a\gamma^\mu \frac{\delta^{(3)}(\vec{p}_3 + \vec{p}_2 - \vec{p}_1)}{\omega(\vec{p}_3) + \omega(\vec{p}_2) - \omega(\vec{p}_1) - i0^+}
$$
$$
\times \Theta(\Delta - |\omega(\vec{p}_3) + \omega(\vec{p}_2) - \omega(\vec{p}_1)|)
$$

$$
\equiv (-ig)T^a\gamma^\mu \frac{\delta^{(3)}(\vec{p}_1 + \vec{p}_2 - \vec{p}_3)}{\omega(\vec{p}_1) + \omega(\vec{p}_2) - \omega(\vec{p}_3) - i0^+}
$$
$$
\times \Theta(\Delta - |\omega(\vec{p}_1) + \omega(\vec{p}_2) - \omega(\vec{p}_3)|)
$$

Figure 4: The diagrammatic rules for vertices.

We form propagator 'like' objects from the spin sums of fermion spinors and an associated energy denominator. Although they are not real propagators in the normal field theory sense of inverted off-shell two-point Green functions, they do represent the transition from one vertex to another. The rules for these are shown in Figure 5, where $\bar{p}^\mu \equiv (p^0, -\vec{p})$.

$$\xrightarrow{\quad p \quad} \equiv \frac{i \, \not{p}}{2\omega(\vec{p})}.$$

$$\mu \overset{\leftarrow p}{\underset{a}{\sim\!\sim\!\sim\!\sim}} \overset{\nu}{b} \equiv \frac{\delta^{ab}}{2\omega(\vec{p})} \left(-g_{\mu\nu} + \frac{p_\mu \overline{p}_\nu + p_\nu \overline{p}_\mu}{(p\overline{p})} \right).$$

Figure 5: The diagrammatic rules for propagators.

As with ordinary field theory we must integrate over all internal momenta and so for each propagator in the asymptotic region we must integrate over its momentum $\int d^{D-1}p/(2\pi)^{D-1}$ in $D-1 = 3 - 2\epsilon$ dimensions. The rules for external particles are exactly the same as for QED or QCD and so do not need to be reproduced here. Finally we must include a factor of $1/n!$ with each diagram, where n represents the order in the coupling in the asymptotic region.

As stated before the soft Møller operators are not necessarily gauge invariant and neither Lorentz invariant. Infrared singularities though will only occur in the region where $\varpi = \sum \sigma_i \omega(\vec{k}_i) = 0$. In this limit Lorentz invariance is restored and so the structure of the singularities will also be Lorentz invariant. Given that our amplitudes will not be gauge invariant, we will perform all calculations including the second term of the gluon propagator which ensures that we sum over physical polarizations only.

4.5 The amplitude $\mathcal{A}(\{q(p_1), \bar{q}(p_2)\}; \gamma)$

Let us start with the amplitude $\mathcal{A}(\{q_{p1}, \bar{q}_{p2}\}; \gamma)$ given in eq. (55). This amplitude consists of four terms and we will look at each of them in turn. The first and second term will be dealt with in Section 4.5.1 and 4.5.2 respectively. For the third term of eq. (55) we need the dressing factor $\mathcal{W}^{(2)}(q_{p1}, \bar{q}_{p2}; q_{q1}, \bar{q}_{q2})$. There are various combinations of interaction terms of the asymptotic Hamiltonian that give rise to non-vanishing contributions to $\mathcal{W}^{(2)}$. These can be found using the diagrammatic rules of Section 4.4. We find that there are four contributing diagrams. These four diagrams can be split into two classes, two self-interaction terms and two one-gluon exchange terms. We consider the former in Section 4.5.3 and the latter in Section 4.5.4. Finally, the last term of eq. (55) will be computed in Section 4.5.5.

26

4.5.1 The Born term

The first term is

$$\int \mathrm{d}\tilde{q}_1 \mathrm{d}\tilde{q}_2 \; \mathcal{W}^{(0)}(q_{p1}, \bar{q}_{p2}; q_{q1}, \bar{q}_{q2}) \times \mathcal{A}^{(0)}(q_{q1}, \bar{q}_{q2}; \gamma(P)) \tag{62}$$

and is of order g^0. As discussed previously, eq. (40), this term corresponds precisely to the tree-level amplitude.

$$
\begin{aligned}
\mathcal{A}^{(0)}(\{q_{p1}, \bar{q}_{p2}\}; \gamma(P)) &= (-ie)\,\delta_{ij}\,\langle p_1|\gamma^\alpha|p_2\rangle\,(2\pi)^D\delta^{(D)}(P - p_1 - p_2) \\
&= \mathcal{A}^{(0)}(q_{p1}, \bar{q}_{p2}; \gamma(P)).
\end{aligned}
$$

We use a notation where $\langle p_i|$ represents the spinor of a massless outgoing fermion with momentum p_i and similarly $|p_j\rangle$ represents the spinor of a massless incoming fermion of momentum p_j. Of course, these spinors depend on the helicity of the fermion, but we suppress this dependence in the notation. The delta function as usual ensures energy-momentum conservation for the process and the δ_{ij} represents the color flow through the diagram.

4.5.2 The virtual term

In the same way we see that the second term of eq. (55) corresponds to the one-loop amplitude

$$
\mathcal{A}^{(2)}(q_{p1}, \bar{q}_{p2}; \gamma(P)) = \tag{63}
$$
$$
C_F \left(\frac{\alpha_s}{2\pi}\right) \left(\frac{\mu^2}{s}\right)^\epsilon \left(-\frac{1}{\epsilon^2} - \frac{3}{2\epsilon} - 4 + \frac{c_R}{2} + \frac{\pi^2}{12}\right) \mathcal{A}^{(0)}(q_{p1}, \bar{q}_{p2}; \gamma(P)).
$$

The infrared singularities appearing in eq. (63) will be cancelled by infrared singularities of the third and fourth term of eq. (55). We should mention that the finite term in eq. (63) depends on the regularization scheme used. The result in conventional dimensional regularization is obtained by setting $c_R = 0$ whereas in dimensional reduction we set $c_R = 1$.

4.5.3 The self-interaction terms

The two self-interaction terms are obtained by taking the interacting terms V_1, V_5 and V_4, V_8 of the asymptotic Hamiltonian as given in eq. (59). Since

there is a symmetry between these two contributions we only need to calculate one of the pair of diagrams. The self interacting term resulting from the vertices V_1, V_5 is shown in Figure 6 and is given by

$$
\begin{aligned}
a_{15}^{\{2,0\}} &\equiv \int d\tilde{q}_1 d\tilde{q}_2 \, g^2 \, \mathcal{W}_{15}^{(2)}(q_{p1}, \bar{q}_{p2}; q_{q1}, \bar{q}_{q2}) \times \mathcal{A}^{(0)}(q_{q1}, \bar{q}_{q2}; \gamma(P)) \qquad (64) \\
&= -\frac{(-ie)\,g^2}{2} \int d^{D-1}q_1 d^{D-1}q_2 \frac{d^{D-1}q_3}{(2\pi)^{D-1}} \, T_{ik}^a T_{kj}^b \\
&\quad \frac{\delta^{ab}}{2\omega(\vec{q}_3)} \left(-g^{\mu\nu} + \frac{q_3^\mu \, \bar{q}_3^\nu + q_3^\nu \, \bar{q}_3^\mu}{(q_3 \bar{q}_3)} \right) \langle p_1 | \gamma_\mu \not{q}_2 \gamma_\nu \not{q}_1 \gamma^\alpha | p_2 \rangle \\
&\quad \frac{\Theta(\Delta - |\omega(\vec{q}_2) + \omega(\vec{q}_3) - \omega(\vec{p}_1)|) \, \Theta(\Delta - |\omega(\vec{q}_2) + \omega(\vec{q}_3) - \omega(\vec{q}_1)|)}{2\omega(\vec{q}_1)\,(\omega(\vec{q}_2) + \omega(\vec{q}_3) - \omega(\vec{p}_1))\, 2\omega(\vec{q}_2)\,(\omega(\vec{q}_2) + \omega(\vec{q}_3) - \omega(\vec{q}_1))} \\
&\quad \delta^{(D-1)}(\vec{q}_2 + \vec{q}_3 - \vec{p}_1)\, \delta^{(D-1)}(\vec{q}_2 + \vec{q}_3 - \vec{q}_1)\, (2\pi)^D \, \delta^{(D)}(P - q_1 - p_2).
\end{aligned}
$$

Note that this expression contains a D-dimensional delta function coming from $\mathcal{A}^{(0)}$ and two $(D-1)$-dimensional delta functions coming from 3-momentum conservation of the vertices in the dressing factor.
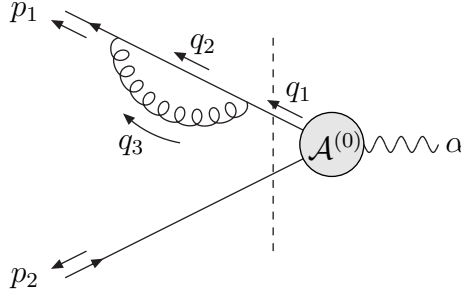

Figure 6: Cut diagram for self interaction with 2-particle intermediate state.

We now proceed to perform the integrals over $\vec{q}_1$ and $\vec{q}_2$, removing the two $(D-1)$ dimensional delta functions. There is an important subtlety here. Since the delta functions are $(D-1)$ dimensional, only the spatial part of the 4-vectors is altered. All 4-vectors in the asymptotic region though must be on-shell and so we are forced to modify the energy component of these 4-vectors to preserve this property. Although these modified 4-vectors are

4 component objects they no longer transform as tensors. This is simply a manifestation of the breaking of Lorentz invariance that occurs in time-ordered perturbation theory. To denote such objects we place curly brackets of the type { } around them, i.e. we define

$$\{p_1 - q_3\} \equiv (\omega(\vec{p}_1 - \vec{q}_3), \vec{p}_1 - \vec{q}_3) \tag{65}$$

We then have

$$a_{15}^{\{2,0\}} = -\frac{(-ie)\,g^2}{2}\,T_{ik}^a T_{kj}^a\,(2\pi)^D \delta^{(D)}(P - p_1 - p_2) \tag{66}$$

$$\int \mathrm{d}\tilde{q}_3 \left(-g^{\mu\nu} + \frac{q_3^\mu\,\overline{q}_3^\nu + q_3^\nu\,\overline{q}_3^\mu}{(q_3 \overline{q}_3)} \right) \Theta(\Delta - |\rho(\vec{q}_3, \vec{p}_1 - \vec{q}_3)|)$$

$$\frac{\langle p_1|\gamma_\mu \{\slashed{p}_1 - \slashed{q}_3\} \gamma_\nu\, \slashed{p}_1 \gamma^\alpha|p_2\rangle}{2\omega(\vec{p}_1) 2\omega(\vec{p}_1 - \vec{q}_3)\rho(\vec{q}_3, \vec{p}_1 - \vec{q}_3)^2}$$

where we defined

$$\rho(\vec{k}_1, \vec{k}_2) \equiv \omega(\vec{k}_1) + \omega(\vec{k}_2) - \omega(\vec{k}_1 + \vec{k}_2) \tag{67}$$

This diagram contains infrared singularities coming from the region where q_3 is soft and/or collinear to p_1. We discuss its evaluation in the Appendix. Multiplying by two to take into account both of the self-interaction diagrams we get the final result

$$2\,a_{15}^{\{2,0\}} = 2 \int \mathrm{d}\tilde{q}_1 \mathrm{d}\tilde{q}_2\, g^2\, \mathcal{W}_{15}^{(2)}(q_{p1}, \overline{q}_{p2}; q_{q1}, \overline{q}_{q2}) \times \mathcal{A}^{(0)}(q_{q1}, \overline{q}_{q2}; \gamma(P)) \tag{68}$$

$$= C_F \left(\frac{\alpha_s}{2\pi} \right) \left(\frac{\mu^2}{s} \right)^\epsilon \left(-\frac{1}{\epsilon^2} - \frac{5}{2\epsilon} + g_1(\Delta) + \frac{c_R}{2} \right.$$

$$\left. + \int \mathrm{d}\tilde{q}_3\, \Theta(\Delta - |\rho(\vec{q}_3, \vec{p}_1 - \vec{q}_3)|) f_1(p_1, p_2, q_3) \right) \mathcal{A}^{(0)}(q_{p1}, \overline{q}_{p2}; \gamma(P))$$

with

$$g_1(\Delta) = -\frac{7}{2} - \frac{5}{2} \left(\frac{\Delta}{2} \right)^2 - \frac{7\pi^2}{12} + \left[\frac{7}{2} + \left(\frac{\Delta}{2} \right) + \frac{1}{2} \left(\frac{\Delta}{2} \right)^2 \right] \log \left(\frac{\Delta}{2} \right)$$

$$+ \log^2 \left(\frac{\Delta}{2} \right) + 2 \log^2 \left(1 + \frac{\Delta}{2} \right) + 4\,\mathrm{Li}_2 \left(\frac{2}{2 + \Delta} \right) \tag{69}$$

and where $f_1(p_1, p_2, q_3)$ is a function that is free from singularities when integrated over $\mathrm{d}\tilde{q}_3$. The explicit form is given in eq. (99). Note that we took care to sum over the physical polarizations of the gluon only and evaluated the diagram in the center-of-mass frame $\vec{p}_1 = -\vec{p}_2$.

29

4.5.4 The one-gluon exchange terms

We now look at the one-gluon exchange diagrams. There are two such diagrams, one for each time ordering of the two vertices. One diagram is obtained from taking the vertices V_1, V_8 of eq. (59), the other from taking the vertices V_4, V_5. These diagrams are symmetric under exchange of all momenta and so we need only calculate one of them. The diagram shown in Figure 7 gives us

$$
\begin{aligned}
a_{18}^{\{2,0\}} &= \int d\widetilde{q}_1 d\widetilde{q}_2 \; g^2 \; \mathcal{W}_{18}^{(2)}(q_{p1}, \bar{q}_{p2}; q_{q1}, \bar{q}_{q2}) \times \mathcal{A}^{(0)}(q_{q1}, \bar{q}_{q2}; \gamma(P)) \qquad (70) \\
&= \frac{(-ie)\,g^2}{2} \int d^{D-1}q_1 d^{D-1}q_2 \frac{d^{D-1}q_3}{(2\pi)^{D-1}} \; T_{ik}^a T_{kj}^b \\
&\quad \frac{\delta^{ab}}{2\omega(\vec{q}_3)} \left(-g^{\mu\nu} + \frac{q_3^\mu\,\bar{q}_3^\nu + q_3^\nu\,\bar{q}_3^\mu}{(q_3\,\bar{q}_3)} \right) \langle p_1| \gamma_\mu \, \slashed{q}_1 \gamma_\alpha \, \slashed{q}_2 \gamma_\nu |p_2\rangle \\
&\quad \frac{\Theta(\Delta - |\omega(\vec{q}_1) + \omega(\vec{q}_3) - \omega(\vec{p}_1)|)\,\Theta(\Delta - |\omega(\vec{p}_2) + \omega(\vec{q}_3) - \omega(\vec{q}_2)|)}{2\omega(\vec{q}_1)\,(\omega(\vec{q}_1) + \omega(\vec{q}_3) - \omega(\vec{p}_1))\; 2\omega(\vec{q}_2)\,(\omega(\vec{p}_2) + \omega(\vec{q}_3) - \omega(\vec{q}_2))} \\
&\quad \delta^{(D-1)}(\vec{q}_1 + \vec{q}_3 - \vec{p}_1)\,\delta^{(D-1)}(\vec{p}_2 + \vec{q}_3 - \vec{q}_2)\,(2\pi)^D\,\delta^{(D)}(P - q_1 - q_2).
\end{aligned}
$$

We again integrate over $\vec{q}_1$ and $\vec{q}_3$ with the delta functions and introduce the on-shell momenta $\{\slashed{p}_1 - \slashed{q}_3\}$ and $\{\slashed{p}_2 + \slashed{q}_3\}$ to obtain

$$
\begin{aligned}
a_{18}^{\{2,0\}} &= \frac{(-ie)\,g^2}{2} T_{ik}^a T_{kj}^a \int \frac{d^{D-1}q_3}{(2\pi)^{D-1}} (2\pi)^D\,\delta^{(D)}(P - \{p_1 - q_3\} - \{p_2 + q_3\}) \\
&\quad \frac{1}{2\omega(\vec{q}_3)} \left(-g^{\mu\nu} + \frac{q_3^\mu\,\bar{q}_3^\nu + q_3^\nu\,\bar{q}_3^\mu}{(q_3\,\bar{q}_3)} \right) \langle p_1| \gamma_\mu \{\slashed{p}_1 - \slashed{q}_3\} \gamma_\alpha \{\slashed{p}_2 + \slashed{q}_3\} \gamma_\nu |p_2\rangle \\
&\quad \frac{\Theta(\Delta - |\rho(\vec{q}_3, \vec{p}_1 - \vec{q}_3)|)\,\Theta(\Delta - |\rho(\vec{q}_3, \vec{p}_2)|)}{2\omega(\vec{p}_1 - \vec{q}_3)\,\rho(\vec{q}_3, \vec{p}_1 - \vec{q}_3)\; 2\omega(\vec{p}_2 + \vec{q}_3)\,\rho(\vec{q}_3, \vec{p}_2)} \qquad (71)
\end{aligned}
$$

Looking at the denominator $\rho(\vec{q}_3, \vec{p}_1 - \vec{q}_3)\,\rho(\vec{q}_3, \vec{p}_2)$ it appears that there are collinear singularities for $q_3 \| p_2$, $q_3 \| p_1$ and a soft singularity $q_3 \to 0$. However, the denominator

$$
\left(-g^{\mu\nu} + \frac{q_3^\mu\,\bar{q}_3^\nu + q_3^\nu\,\bar{q}_3^\mu}{(q_3\,\bar{q}_3)} \right) \langle p_1| \gamma_\mu \{\slashed{p}_1 - \slashed{q}_3\} \gamma_\alpha \{\slashed{p}_2 + \slashed{q}_3\} \gamma_\nu |p_2\rangle \qquad (72)
$$

vanishes in the collinear regions $q_3 \| p_2$ and $q_3 \| p_1$. Thus, this diagram has only a soft singularity.
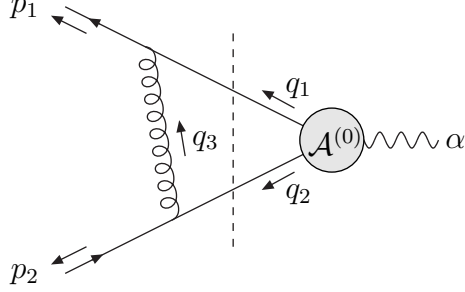
Figure 7: Cut diagram for the 2-particle cut diagram with one-gluon exchange in the asymptotic region.

We delegate the explicit evaluation of $a_{18}^{\{2,0\}}$ to the Appendix. Multiplying by two to account for both one-gluon exchange diagrams we have the final result

$$
\begin{aligned}
2\, a_{18}^{\{2,0\}} &= 2 \int \mathrm{d}\tilde{q}_1 \mathrm{d}\tilde{q}_2 \; g^2 \; \mathcal{W}_{18}^{(2)}(q_{p1}, \bar{q}_{p2}; q_{q1}, \bar{q}_{q2}) \times \mathcal{A}^{(0)}(q_{q1}, \bar{q}_{q2}; \gamma(P)) \quad (73) \\
&= C_F \left(\frac{\alpha_s}{2\pi}\right) \left(\frac{\mu^2}{s}\right)^{\epsilon} \left(\left(\frac{1}{\epsilon} + g_2(\Delta)\right) \mathcal{A}^{(0)}(q_{p1}, \bar{q}_{p2}; \gamma(P)) \right. \\
&\quad - (-ie)\, \delta_{ij} \, \langle p_1 | \gamma^\alpha | p_2 \rangle (2\pi)^{(D-1)} \delta^{(D-1)}(\vec{P} - \vec{p}_1 - \vec{p}_2) \\
&\quad \left. \times \int \mathrm{d}\tilde{q}_3 \; \Theta(\Delta - |\rho(\vec{q}_3, \vec{p}_1 - \vec{q}_3)|)\Theta(\Delta - |\rho(\vec{q}_3, \vec{p}_2)|) f_2(p_1, p_2, q_3) \right)
\end{aligned}
$$

where again we have not performed the finite f_2 integral analytically and

$$
g_2(\Delta) = 2\log 2 - 2\log\left(\frac{\Delta}{2}\right). \quad (74)
$$

The explicit form of f_2 is given in eq. (102).

4.5.5   3 Particle Cut Diagram

Let us now turn to the forth part of eq. (55). For this term we need the dressing factor $\mathcal{W}^{(1)}(q_{p1}, \bar{q}_{p2}; q_{q1}, \bar{q}_{q2}, g_{q3})$. Again we use the diagrammatic rules of Section 4.4.
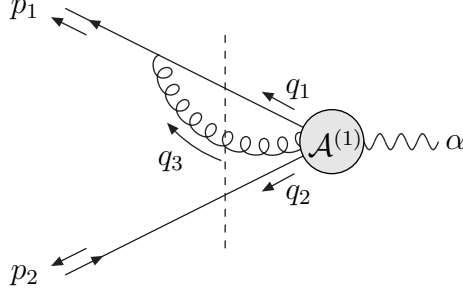
Figure 8: Cut diagram for 3-particle intermediate state.

There are two possible diagrams as the gluon can be absorbed either by the quark or antiquark line. The two diagrams are obtained by taking either the vertex V_1 or V_4 and they are symmetric under exchange of momenta. So we need only calculate one of them. For the diagram shown in Figure 8 we get

$$
\begin{aligned}
a_1^{\{1,1\}} &= \int d\tilde{q}_1 d\tilde{q}_2 d\tilde{q}_3 \, g^2 \, \mathcal{W}_1^{(1)}(q_{p1}, \bar{q}_{p2}; q_{q1}, \bar{q}_{q2}, g_{q3}) \times \mathcal{A}^{(1)}(q_{q1}, \bar{q}_{q2}, g_{q3}; \gamma(P)) \\
&= (-ie) \, g^2 \int d^{D-1}q_1 d^{D-1}q_2 \frac{d^{D-1}q_3}{(2\pi)^{D-1}} \frac{\delta^{ab}}{2\omega(\vec{q}_3)} \, T_{ik}^a T_{kj}^b \qquad (75) \\
&\quad \left(-g^{\mu\nu} + \frac{q_3^\mu \, \overline{q}_3^\nu + q_3^\nu \, \overline{q}_3^\mu}{(q_3 \, \overline{q}_3)} \right) \frac{\Theta(\Delta - |\omega(\vec{q}_1) + \omega(\vec{q}_3) - \omega(\vec{p}_1)|)}{2\omega(\vec{q}_1) \, (\omega(\vec{q}_1) + \omega(\vec{q}_3) - \omega(\vec{p}_1))} \\
&\quad \langle p_1 | \gamma_\mu \, \slashed{q}_1 \left(\frac{\gamma^\nu (\slashed{q}_1 + \slashed{q}_3) \gamma^\alpha}{2(q_1 q_3)} - \frac{\gamma^\alpha (\slashed{q}_2 + \slashed{q}_3) \gamma^\nu}{2(q_2 q_3)} \right) | p_2 \rangle \\
&\quad \delta^{(D-1)}(\vec{q}_1 + \vec{q}_3 - \vec{p}_1) \, \delta^{(D-1)}(\vec{p}_2 - \vec{q}_2)(2\pi)^D \, \delta^{(D)}(P - q_1 - q_2).
\end{aligned}
$$

After integration over $\vec{q}_1$ and $\vec{q}_2$ using the delta functions we observe that there are collinear singularities $q_3 \| p_1$ and soft singularities $q_3 \to 0$. There are, however, no collinear singularities $q_3 \| p_2$. This is expected since the amplitude $\mathcal{A}^{(1)}(q_{q1}, \bar{q}_{q2}, g_{q3}; \gamma(P))$ has only an integrable square-root singularity for $q_3 \| q_2$ and the dressing factor $\mathcal{W}_1^{(1)}(q_{p1}, \bar{q}_{p2}; q_{q1}, \bar{q}_{q2}, g_{q3})$ is regular for $q_3 \| q_2$.

As for the other diagrams we have to multiply by two to take into account

32

both pairs of diagrams and we get the final result

$$
\begin{aligned}
2\, a_1^{\{1,1\}} \;=\;& 2 \int \mathrm{d}\widetilde{q}_1 \mathrm{d}\widetilde{q}_2 \mathrm{d}\widetilde{q}_3 \, g^2 \, \mathcal{W}_1^{(1)}(q_{p1}, \bar{q}_{p2}; q_{q1}, \bar{q}_{q2}, g_{q3}) \hspace{2cm} (76)\\
& \hspace{3cm} \times \; \mathcal{A}^{(1)}(q_{q1}, \bar{q}_{q2}, g_{q3}; \gamma(P))\\
\;=\;& C_F \left(\frac{\alpha_s}{2\pi}\right)\left(\frac{\mu^2}{s}\right)^\epsilon \left(\left(\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+g_3(\Delta)-c_R\right)\mathcal{A}^{(0)}(q_{p1},\bar{q}_{p2};\gamma(P))\right.\\
& +\;(-ie)\,\delta_{ij}\,\langle p_1|\gamma^\alpha|p_2\rangle (2\pi)^{(D-1)}\delta^{(D-1)}(\vec{P}-\vec{p}_1-\vec{p}_2)\\
& \hspace{2cm} \left. \times \int \mathrm{d}\widetilde{q}_3\,\Theta(\Delta-|\rho(\vec{q}_3,\vec{p}_1-\vec{q}_3)|)f_3(p_1,p_2,q_3)\right)
\end{aligned}
$$

where

$$
\begin{aligned}
g_3(\Delta) \;=\;& 7+\left(\frac{\Delta}{2}\right)^2+\frac{7\pi^2}{6}+\left[-3+2\left(\frac{\Delta}{2}\right)-\left(\frac{\Delta}{2}\right)^2\right]\log\left(\frac{\Delta}{2}\right)\\
& -\;2\log^2\left(\frac{\Delta}{2}\right)-4\log^2\left(1+\frac{\Delta}{2}\right)-8\,\mathrm{Li}_2\left(\frac{2}{2+\Delta}\right). \hspace{1cm} (77)
\end{aligned}
$$

The function f_3 is given in eq. (104) and does not produce any infrared singularity upon integration over $\mathrm{d}\widetilde{q}_3$.

4.5.6 An infrared-finite amplitude

We have now calculated all terms contributing to the amplitude $\mathcal{A}(\{q,\bar{q}\};\gamma)$, eq. (55), at next-to-leading order. Using eqs. (63, 68, 73) and (76) to assemble the amplitude we get

$$
\mathcal{A}(\{q_{p1},\bar{q}_{p2}\};\gamma) = \hspace{8cm} (78)
$$

$$
\begin{aligned}
& 1+C_F\left(\frac{\alpha_s}{2\pi}\right)\left(g_1(\Delta)+g_2(\Delta)+g_3(\Delta)-4+\frac{\pi^2}{12}\right)\mathcal{A}^{(0)}(q_{p1},\bar{q}_{p2};\gamma(P))\\
& +(-ie)\,\delta_{ij}\,\langle p_1|\gamma^\alpha|p_2\rangle(2\pi)^{(D-1)}\delta^{(D-1)}(\vec{P}-\vec{p}_1-\vec{p}_2)\\
& \times \int \mathrm{d}\widetilde{q}_3\,\left(f_1(p_1,p_2,q_3)\Theta(\Delta-|\rho(\vec{q}_3,\vec{p}_1-\vec{q}_3)|)\delta(\sqrt{s}-\omega(\vec{p}_1)-\omega(\vec{p}_2))\right.\\
& \hspace{2cm} +f_2(p_1,p_2,q_3)\Theta(\Delta-|\rho(\vec{q}_3,\vec{p}_1-\vec{q}_3)|)\Theta(\Delta-|\rho(\vec{q}_3,\vec{p}_2)|)\\
& \hspace{2cm} \left. +f_3(p_1,p_2,q_3)\Theta(\Delta-|\rho(\vec{q}_3,\vec{p}_1-\vec{q}_3)|)\right)
\end{aligned}
$$

up to order α_s in the coupling. The functions g_1, g_2, g_3 are given in eqs. (69, 74) and (77) and the functions f_1, f_2 and f_3 are given in eqs. (99, 102) and (104) respectively.

We see that this result is completely free of infrared singularities. We are only left with some finite Δ dependent terms, g_i and some finite terms, f_i which will in general need to be numerically integrated. Even though the amplitude $\mathcal{A}(\{q_{p1}, \bar{q}_{p2}\}; \gamma)$ depends on Δ this dependence will disappear when we combine the various amplitudes to calculate physical observables.

4.6   The amplitude $\mathcal{A}(\{q(p_1), \bar{q}(p_2), g(p_3)\}; \gamma)$

We are now going to calculate the amplitude $\mathcal{A}(\{q_{p1}, \bar{q}_{p2}, g_{p3}\}; \gamma)$ given in eq. (56). There are only two terms to calculate for this amplitude and there is no integration over the final state gluon as it is now a real final state particle.
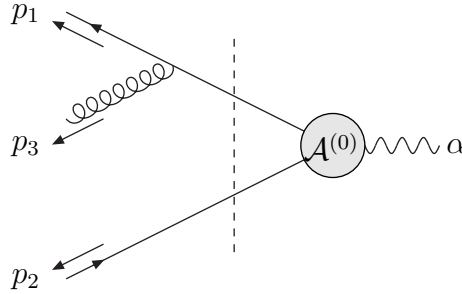

Figure 9: Cut diagram for the 3-particle asymptotic region with a 2-particle intermediate state.

Again we calculate these terms using the diagrammatic rules from Section 4.4. Let us start with the diagrams where the gluon is emitted in the dressing factor. Figure 9 shows one of the two possible diagrams, the other is exactly the same but with all momenta interchanged. So for both diagrams

we have

$$
\int \mathrm{d}\tilde{q}_1 \mathrm{d}\tilde{q}_2 \; g \, \mathcal{W}_1^{(1)}(q_{p1}, \bar{q}_{p2}, g_{p3}; q_{q1}, \bar{q}_{q2}) \times \mathcal{A}^{(0)}(q_{q1}, \bar{q}_{q2}; \gamma(P))
$$

$$
= \quad (-ie) g \, T_{ij}^a \, (2\pi)^D \delta^{(D-1)}(\vec{P} - \vec{p}_1 - \vec{p}_2 - \vec{p}_3) \, \langle p_1| \tag{79}
$$

$$
\Bigg(- \frac{\slashed{\epsilon}_{p3}\{\slashed{p}_1 + \slashed{p}_3\}\gamma^\alpha}{2\omega(\vec{p}_1 + \vec{p}_3)r_1}\Theta(\Delta - |r_1|)\delta(\sqrt{s} - \omega(\vec{p}_1) - \omega(\vec{p}_2) - \omega(\vec{p}_3) + r_1)
$$

$$
+ \frac{\gamma^\alpha\{\slashed{p}_2 + \slashed{p}_3\}\slashed{\epsilon}_{p3}}{2\omega(\vec{p}_2 + \vec{p}_3)r_2}\Theta(\Delta - |r_2|)\delta(\sqrt{s} - \omega(\vec{p}_1) - \omega(\vec{p}_2) - \omega(\vec{p}_3) + r_2)\Bigg)|p_2\rangle
$$

where we used the notation

$$
\begin{aligned}
r_1 &\equiv \rho(\vec{p}_1, \vec{p}_3) = \omega(\vec{p}_1) + \omega(\vec{p}_3) - \omega(\vec{p}_1 + \vec{p}_3) \\
r_2 &\equiv \rho(\vec{p}_2, \vec{p}_3) = \omega(\vec{p}_2) + \omega(\vec{p}_3) - \omega(\vec{p}_2 + \vec{p}_3),
\end{aligned} \tag{80}
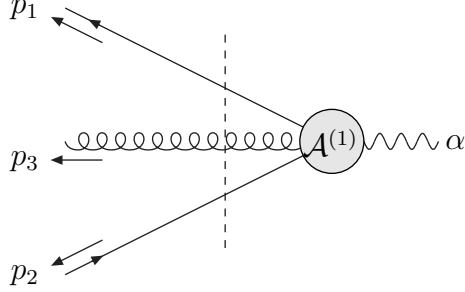$$

with ρ defined in eq. (67).

Figure 10: Cut diagram for 3-particle asymptotic region with a 3-particle intermediate state.

The second contribution is just the usual $\mathcal{A}^{(1)}(q_{p1}, \bar{q}_{p2}, g_{p3}; \gamma(P))$ amplitude. The three external particles of this amplitude do not interact in the asymptotic region and so we simply have the diagram as shown in Figure 10, this gives

$$\int d\tilde{q}_1 \, d\tilde{q}_2 \, g \, \mathcal{W}^{(0)}(q_{p1}, \bar{q}_{p2}, g_{p3}; q_{q1}, \bar{q}_{q2}, g_{q3}) \times \mathcal{A}^{(1)}(q_{q1}, \bar{q}_{q2}, g_{q3}; \gamma(P))$$

$$= (-ie)g \, T_{ij}^a \, \langle p_1| \left(\frac{\not{\epsilon}_{p3}(\not{p}_1 + \not{p}_3)\gamma^\alpha}{2(p_1 p_3)} - \frac{\gamma^\alpha(\not{p}_2 + \not{p}_3)\not{\epsilon}_{p3}}{2(p_2 p_3)} \right) |p_2\rangle$$

$$(2\pi)^D \delta^{(D)}(P - p_1 - p_2 - p_3). \tag{81}$$

We now assemble eq. (56) to find

$$\mathcal{A}(\{q_{p1}, \bar{q}_{p2}, g_{p3}\}; \gamma) = (-ie)g \, T_{ij}^a \, \langle p_1| \Bigg($$

$$-\frac{\not{\epsilon}_{p3}\{\not{p}_1 + \not{p}_3\}\gamma^\alpha}{2\omega(\vec{p}_1 + \vec{p}_3)r_1} \Theta(\Delta - |r_1|)\delta(\sqrt{s} - \omega(\vec{p}_1) - \omega(\vec{p}_2) - \omega(\vec{p}_3) + r_1)$$

$$+\frac{\not{\epsilon}_{p3}(\not{p}_1 + \not{p}_3)\gamma^\alpha}{2(p_1 p_3)}\delta(\sqrt{s} - \omega(\vec{p}_1) - \omega(\vec{p}_2) - \omega(\vec{p}_3))$$

$$+\frac{\gamma^\alpha\{\not{p}_2 + \not{p}_3\}\not{\epsilon}_{p3}}{2\omega(\vec{p}_2 + \vec{p}_3)r_2} \Theta(\Delta - |r_2|)\delta(\sqrt{s} - \omega(\vec{p}_1) - \omega(\vec{p}_2) - \omega(\vec{p}_3) + r_2)$$

$$-\frac{\gamma^\alpha(\not{p}_2 + \not{p}_3)\not{\epsilon}_{p3}}{2(p_2 p_3)}\delta(\sqrt{s} - \omega(\vec{p}_1) - \omega(\vec{p}_2) - \omega(\vec{p}_3)) \Bigg) |p_2\rangle$$

$$(2\pi)^D \delta^{(D-1)}(\vec{P} - \vec{p}_1 - \vec{p}_2 - \vec{p}_3). \tag{82}$$

This amplitude splits up into two pairs. The first (last) two terms are due to the gluon being emitted from the leg p_1 (p_2). Looking at the first two terms shows that for $r_1 > \Delta$ the contribution from the asymptotic region disappears. We are then left with the normal amplitude, $\mathcal{A}(q_{p1}, \bar{q}_{p2}, g_{p3}; \gamma)$. For $r_1 < \Delta$ the term from the asymptotic region does contribute and will cancel any potential infrared singularities. We can see this by taking the limit $\Delta \to 0$, we have

$$\omega(\vec{p}_1 + \vec{p}_3)r_1 \rightarrow (p_1 p_3),$$

$$\{\not{p}_1 + \not{p}_3\} \rightarrow (\not{p}_1 + \not{p}_3),$$

$$\delta(\sqrt{s} - \omega(\vec{p}_1) - \omega(\vec{p}_2) - \omega(\vec{p}_3) + r_1) \rightarrow \delta(\sqrt{s} - \omega(\vec{p}_1) - \omega(\vec{p}_2) - \omega(\vec{p}_3)).$$

With these we can see that the terms from the asymptotic region approach those of the normal amplitude in the soft and collinear limits, but with the opposite sign. So the two terms will cancel in the $\Delta \to 0$ limit, leaving us with an amplitude that is infrared finite when integrated over the phase space.

36

4.7 Calculation of the total cross section

In the previous sections we computed the two infrared finite amplitudes that contribute to the process $\gamma^*(P) \to 2$ jets at next-to-leading order. In this section we would like to check our results by computing the total cross section, starting from the infrared finite amplitudes, eqs. (78) and (82). Of course, we have to recover the well known result, eq. (54).

Let us stress that the idea of our approach is to compute the amplitudes numerically and perform the phase-space integration also numerically. It is for the sole purpose of checking our results and facilitating the comparison with eq. (54) that in this section we compute the total cross section analytically.

Usually a non zero value of Δ would be chosen for a numerical calculation and we would expect all Δ dependence to cancel between the contributions of the two amplitudes (squared) to the cross section. Here though to simplify the analytical calculation we will take the limit $\Delta \to 0$. In this limit even the infrared-finite amplitudes are proportional to a four-dimensional delta function and we can use the standard procedure to obtain the total cross section from the amplitudes. However, since the amplitudes are singular for $\Delta \to 0$ we must be careful in taking this limit and leave it until the end of the calculation. The f_n finite terms of eq. (78) which we would usually have to calculate numerically will all go to zero in this limit. This simplification occurs because the region of integration shrinks to zero as $\Delta \to 0$ and as these terms are finite they can no longer give a contribution.

We now use our infrared finite amplitudes eq. (55) and eq. (56) instead of eq. (45) and eq. (46) and square them in the usual way to obtain

$$\sigma = \sigma_{\{q\bar{q}\}} + \sigma_{\{q\bar{q}g\}}, \tag{83}$$

where

$$\sigma_{\{q\bar{q}\}} = \int d\Phi_2 \left| \mathcal{A}(\{q_{p1}, \bar{q}_{p2}\}; \gamma) \right|^2, \tag{84}$$

$$\sigma_{\{q\bar{q}g\}} = \int d\Phi_3 \left| \mathcal{A}(\{q_{p1}, \bar{q}_{p2}, g_{p3}\}; \gamma) \right|^2. \tag{85}$$

Here we integrate eq. (84) over the two particle phase space and eq. (85) over the three particle phase space.

First we rewrite the three-particle final state amplitude, eq. (82), in a more convenient form

$$
\mathcal{A}(\{q_{p1}, \bar{q}_{p2}, g_{p3}\}; \gamma) \;=\; (-ie)g\, T^a_{ij}\, \langle p_1 | \Bigg(
$$

$$
\left(-\frac{\slashed{\epsilon}_{p3}\{\slashed{p}_1 + \slashed{p}_3\}\gamma^\alpha}{2\omega(\vec{p}_1 + \vec{p}_3)r_1}\delta(E + r_1) + \frac{\slashed{\epsilon}_{p3}(\slashed{p}_1 + \slashed{p}_3)\gamma^\alpha}{2(p_1 p_3)}\delta(E) \right)\Theta(\Delta - |r_1|)
$$

$$
+\frac{\slashed{\epsilon}_{p3}(\slashed{p}_1 + \slashed{p}_3)\gamma^\alpha}{2(p_1 p_3)}\delta(E)\Theta(|r_1| - \Delta)
$$

$$
+\left(\frac{\gamma^\alpha\{\slashed{p}_2 + \slashed{p}_3\}\,\slashed{\epsilon}_{p3}}{2\omega(\vec{p}_2 + \vec{p}_3)r_2}\delta(E + r_2) - \frac{\gamma^\alpha(\slashed{p}_2 + \slashed{p}_3)\,\slashed{\epsilon}_{p3}}{2(p_2 p_3)}\delta(E) \right)\Theta(\Delta - |r_2|)
$$

$$
-\frac{\gamma^\alpha(\slashed{p}_2 + \slashed{p}_3)\,\slashed{\epsilon}_{p3}}{2(p_2 p_3)}\delta(E)\Theta(|r_2| - \Delta) \Bigg)|p_2\rangle
$$

$$
(2\pi)^D\delta^{(D-1)}(\vec{P} - \vec{p}_1 - \vec{p}_2 - \vec{p}_3), \tag{86}
$$

where $E = \sqrt{s} - \omega(\vec{p}_1) - \omega(\vec{p}_2) - \omega(\vec{p}_3)$. Taking eq. (86) we then square it in the usual way and sum over the gluon polarizations using

$$
\sum \epsilon^\mu_{p3}\epsilon^\nu_{p3} = -g^{\mu\nu} + \frac{p^\mu_3\bar{p}^\nu_3 + p^\nu_3\bar{p}^\mu_3}{(p_3\bar{p}_3)}
$$

where $\bar{p}_3 = (\omega(\vec{p}_3), -\vec{p}_3)$. This is because the amplitude is no longer gauge invariant as we are using dressed states. At this point we drop any terms multiplied by $\Theta(\Delta - |r_1|)$ or $\Theta(\Delta - |r_2|)$. These terms are finite and therefore can be shown to go to zero in the $\Delta \to 0$ limit after we have performed the three particle phase space integral in a similar way to the f_n terms.

After integrating one of the phase space integrals using the delta function we are left with

$$
|\mathcal{A}(\{q_{p1}, \bar{q}_{p2}, g_{p3}\}; \gamma)|^2 = 4\, C_F\, (2\pi)^{3-2D} \int \frac{d\Omega_{D-1}}{2^{D-1}}d\Omega_{D-2} \tag{87}
$$

$$
\Bigg(\int_0^{\frac{\Delta}{2}} dy_{13} \int_{1-y_{13}}^0 dy_{23}\frac{2y_{23} - y_{13}\left(y_{13} + y_{23}\right)^2}{y_{13}\left(y_{13} + y_{23}\right)^2}
$$

$$
-\int_0^{1-\frac{\Delta}{2}} dy_{13} \int_{1-y_{13}}^{\frac{\Delta}{2}} dy_{23}\,\frac{y_{23}^3 + y_{13}^2 y_{23} + 2y_{13}\left(y_{23} - 1\right)^2}{y_{23}\left(y_{13} + y_{23}\right)^2}
$$

$$
+\int_{\frac{\Delta}{2}}^{1-\frac{\Delta}{2}} dy_{13} \int_{1-y_{13}}^{\frac{\Delta}{2}} dy_{23}\left(2 - \frac{2 - y_{23}}{y_{13}} - \frac{2 - y_{13}}{y_{23}} + \frac{4}{\left(y_{13} + y_{23}\right)^2} \right)\Bigg)
$$

where we defined

$$y_{ij} \equiv \frac{2(p_i p_j)}{\xi_{p_1}^2}. \tag{88}$$

We perform the final two integrals and then prematurely take the $\Delta \to 0$ limit everywhere except in the $\log(\Delta)$ terms, as these diverge in this limit. The $\log(\Delta)$ terms will cancel later in the final result. This then leaves

$$\frac{1}{\sigma_0}\sigma_{\{q\bar{q}g\}} = C_F \left(\frac{\alpha_s}{\pi}\right)\left(\frac{5}{4} - \log 4 + \frac{3}{2}\log\left(\frac{\Delta}{2}\right) + \log^2\left(\frac{\Delta}{2}\right)\right) \tag{89}$$

where σ_0 is the total Born cross section as given in eq. (48).

Now we calculate $|\mathcal{A}(\{q_{p1}, \bar{q}_{p2}\}; \gamma)|^2$. Again we take the $\Delta \to 0$ limit early except for the $\log(\Delta)$ pieces of the g_n terms in the finite part of eq. (78). As stated before the $f_n(p_1, p_2, q_3)$ terms go to zero and so we have,

$$|\mathcal{A}(\{q_{p1}, \bar{q}_{p2}\}; \gamma)|^2 = \left|\mathcal{A}^{(0)}(q_{p1}, \bar{q}_{p2}; \gamma(P))\right|^2 \tag{90}$$

$$\times \ \left(1 + C_F\left(\frac{\alpha_s}{2\pi}\right)\left(-\frac{1}{2} + \log 4 - \frac{3}{2}\log\left(\frac{\Delta}{2}\right) - \log^2\left(\frac{\Delta}{2}\right)\right)\right)^2$$

After integrating over the two particle phase space we get

$$\frac{\sigma_{\{q\bar{q}\}}}{\sigma_0} = \left(1 + C_F\frac{\alpha_s}{\pi}\left(-\frac{1}{2} + \log 4 - \frac{3}{2}\log\left(\frac{\Delta}{2}\right) - \log^2\left(\frac{\Delta}{2}\right)\right) + \mathcal{O}(\alpha_s^2)\right) \tag{91}$$

Putting eq. (89) and eq. (91) together gives finally

$$\frac{\sigma}{\sigma_0} = \left(1 + \left(\frac{\alpha_s}{\pi}\right)\frac{3}{4}C_F + \mathcal{O}(\alpha_s^2)\right). \tag{92}$$

We have recovered the well known result for the total $\gamma \to q\bar{q}$ cross section and all the Δ dependence of the amplitudes has disappeared including the $\log(\Delta)$ terms, justifying our taking of the $\Delta \to 0$ limit early.

5 Summary and outlook

We have presented a method on how to construct infrared finite amplitudes and applied it to the case of $e^+e^- \to 2$ jets at next-to-leading order in the

strong coupling. The idea is to separate from the Hamiltonian a part that describes the asymptotic dynamics. This asymptotic Hamiltonian is then used to asymptotically evolve the usual states of the Fock space. In this way we construct dressed states, eqs. (23) and (24), such that the transition amplitudes between these states are free from infrared singularities.

Contrary to most of the previous work done in this field we are not so much interested in obtaining all-order resumed results taking into account soft emission of an arbitrary number of gauge bosons from external partons. Our aim is to construct dressed states explicitly order-by order in perturbation theory and use them to do explicit calculations. In this paper we have done this for a particularly simple final state up to next-to-leading order. In the future we would like to expand this to more complicated external states and higher orders.

The reason that we cannot obtain all-order results is that we include the collinear singularities as well. In non-abelian theories these singularities cannot be avoided. The additional complications due to the collinear singularities make it impossible to obtain exact solutions to the asymptotic dynamics. Collinear singularities have been considered previously [12, 13, 14] but to the best of our knowledge the amplitudes presented in this work are the first infrared-finite amplitudes for a realistic scattering process in QCD.

As for the standard approach, physical cross sections obtain in general contributions from more than one partonic process. However, in our case all these contributions are separately finite. They depend on a parameter, Δ, that determines the precise split of the Hamiltonian into an asymptotic Hamiltonian and the remainder. The result for any physical quantity is independent of this parameter as long as it is smaller than any experimental resolution. For any finite value of Δ the amplitude contains a part that is not proportional to an energy conserving delta function which represents the spread of the initial wave packet due to the asymptotic evolution.

For any physical cross section at any order in perturbation theory we will get the same answer using the standard cross-section method or infrared-finite amplitudes. Thus, one might wonder what has been gained using this approach. Apart from the conceptional benefit that the S-matrix between dressed states is well defined there are also practical advantages. First of all, the avoidance of infrared singularities facilitates the use of numerical methods. This might not be apparent in the approach we have taken. In

fact, using eq. (28) to split the infrared finite amplitudes into separately divergent factors still requires us to use an infrared regulator (dimensional regularization in our case) and revert to analytical calculations. However, since the final amplitude is infrared finite it is feasible to compute it directly in a numerical way, avoiding the split into separately divergent pieces. Once the amplitudes have been obtained, the integration over the phase space is trivial and no sophisticated method is needed. This also opens up the possibility of combining fixed-order calculations directly with a parton shower approach.

Needless to say that the explicit example we considered, $e^+e^- \rightarrow 2$ jets has many simplifying features. To start with, the non-abelian nature of QCD does not really enter. Secondly, we only considered the amplitudes at next-to-leading order. Furthermore, the initial state does not interact strongly.

The last point simply results in the fact that there is no need to dress the initial state. While this is a simplification concerning the amount of computations to be performed, there is no conceptual problem associated with more complicated initial states. If the initial state contains hadrons a physical cross section is obtained by folding the partonic cross section with parton densities. In the conventional approach these parton densities are associated with the probability of finding a certain partonic state within a hadron. In our case, we would have to use modified parton densities that are related to the probability of finding a certain dressed state within a hadron. Thus the global analyzes of extracting the parton densities would have to be modified and repeated.

The fact that the non-abelian nature of QCD does not really show up in the explicit example we considered results in a particularly simple asymptotic Hamiltonian. In fact, the asymptotic Hamiltonian we use involves only quark-gluon interactions and is basically the same that was used many times previously [14]. Again, this results in a technical simplification of the computation and facilitates the explicit construction of the asymptotic Hamiltonian. In more complicated examples the full non-abelian structure of the asymptotic Hamiltonian will enter the problem and its construction will be much more involved. However, the only crucial feature is that the asymptotic Hamiltonian reproduces the full asymptotic dynamics, i.e. it has to reproduce the soft and collinear behavior of the full theory. There are no further requirements and the construction of dressed states presented in this paper can be taken over directly. However, it is clear that the construction

41

used so far is rather cumbersome. In order to exploit the advantage of the infrared finiteness a systematic numerical approach should be developed. This will become particularly important if the method is to be extended beyond next-to-leading order.

Appendix

In this Appendix we give some details concerning the evaluation of the diagrams mentioned in Section 4.5.

We consider first with the self-interaction term $a_{15}^{\{2,0\}}$ of Section 4.5.3. We start from eq. (66), substitute $\not{p}_1 - \not{q}_3 + \not{r}$ for $\{\not{p}_1 - \not{q}_3\}$, where $r = (r_0, \vec{0})$ with $r_0 = \rho(\vec{q}_3, \vec{p}_1 - \vec{q}_3)$ and then expand the numerator to obtain

$$
\begin{aligned}
a_{15}^{\{2,0\}} &= \frac{(-ie)\,g^2}{4} T_{ik}^a T_{kj}^a\, (2\pi)^D \delta^{(D)}(P - p_1 - p_2) \int d\widetilde{q}_3\; \Theta(\Delta - |r_0|) \quad (93) \\
&\left((D-2)((p_1 q_3) - (p_1 r)) - \frac{4(p_1 q_3)(p_1 \overline{q}_3)}{(q_3 \overline{q}_3)} - \frac{2r_0 (p_1 q_3)(p_2 q_3)}{\omega(\vec{p}_1)(q_3 \overline{q}_3)} \right) \\
&\frac{\langle p_1 | \gamma^\alpha | p_2 \rangle}{\omega(\vec{p}_1)\, \omega(\vec{p}_1 - \vec{q}_3)\, r_0^2}
\end{aligned}
$$

This expression contains infrared singularities coming from the region where q_3 is soft and/or collinear to p_1. In order to evaluate the expression, eq. (93) we choose to parameterize the momenta in the center-of-mass frame. The momenta are all on-shell and are defined as

$$
\begin{aligned}
P &= \sqrt{s}(1, \mathbf{0}, 0), \quad (94) \\
p_1 &= \frac{\xi_{p_1}}{2}(1, \mathbf{0}, 1), \\
p_2 &= \frac{\xi_{p_1}}{2}(1, \mathbf{0}, -1), \\
q_3 &= \frac{\xi_{p_1}}{2}\, z\, (1, \sqrt{1-y^2}\, \mathbf{e}_T, y), \\
\{p_1 - q_3\} &= \frac{\xi_{p_1}}{2}(\sqrt{1 - 2zy + z^2}, -z\sqrt{1-y^2}\, \mathbf{e}_T, 1 - zy),
\end{aligned}
$$

where $\mathbf{0}$ is the null vector in a $(2-2\epsilon)$-dimensional space, $\mathbf{e}_T$ is a unit vector in the $(2-2\epsilon)$-dimensional transverse momentum space and we have $0 \leq z \leq \infty$,

$-1 \leq y \leq 1$. The singular limits are then given by the limits $z \to 0$ for soft singularities, $y \to 1$ for $q_3 \| p_1$ singularities and $y \to -1$ for $q_3 \| p_2$ singularities. As the asymptotic region does not conserve energy we find that the upper limit of z goes to ∞. This would suggest the possibility of UV singularities in the asymptotic regions. However we will see that the Θ function will restrict this upper limit to a finite value, removing the need to renormalize these regions. The integral measure is given by

$$\int \mathrm{d}\widetilde{q}_3 \, \Theta(\Delta - r_0) \to \tag{95}$$
$$\left(\frac{\mu^2}{s}\right)^\epsilon \frac{1}{2(2\pi)^{3-2\epsilon}} \frac{\xi_{p_1}^2}{4} z^{1-2\epsilon} (1-y)^{-\epsilon} (1+y)^{-\epsilon} \, dy \, dz \, d\Omega_{(2-2\epsilon)}$$

where we have three separate integration regions for the z and y integrals,

$$0 \leq z \leq \frac{2}{\Delta} \frac{(2+\Delta)}{(1-y+\Delta)} \quad \text{with} \quad -1 \leq y \leq \frac{2-\Delta^2}{2},$$

$$0 \leq z \leq 1 \quad \text{with} \quad \frac{2-\Delta^2}{2} \leq y \leq 1,$$

$$1 \leq z \leq \frac{2}{\Delta} \frac{(2+\Delta)}{(1-y+\Delta)} \quad \text{with} \quad \frac{2-\Delta^2}{2} \leq y \leq 1.$$

The infrared singularities are in the first two regions whereas the last region will give a finite contribution. The remaining angular integral is given by

$$\int d\Omega_{(2-2\epsilon)} = \frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}. \tag{96}$$

We now turn back to eq. (93) and notice that the infrared singularities q_3 soft and/or collinear to p_1 come from the region $z = 0$ and $y = 1$ but not $y = -1$. We use the subtraction method to isolate these singularities and evaluate them analytically. Writing the integrand schematically as a function $F(z, y)$ we write

$$\begin{aligned} F(z,y) &= (F(0,y) + F(z,1) - F(0,1)) \\ &+ (F(z,y) - F(0,y) - F(z,1) + F(0,1)) . \end{aligned} \tag{97}$$

The first term contains all the divergent pieces whereas the second term will give a finite contribution upon integration over $\mathrm{d}\widetilde{q}_3$. Applying this method

43

to eq. (93) we obtain

$$
a_{15}^{\{2,0\}} = (-ie)\, g^2\, T_{ik}^a T_{kj}^a \left(\frac{\mu^2}{s}\right)^\epsilon \frac{1}{2(2\pi)^{3-2\epsilon}}\, (2\pi)^D \delta^{(D)}(\vec{P} - \vec{p}_1 - \vec{p}_2) \quad (98)
$$

$$
\langle p_1|\gamma^\alpha|p_2\rangle \int d\Omega_{(2-2\epsilon)}\, dy\, dz\, z^{1-2\epsilon}\, (1-y)^{-\epsilon}\, (1+y)^{-\epsilon}
$$

$$
\times \left(\frac{(2-D)}{4(1-y)} + \frac{(2z-1-y)}{2(1-y)z^2} + f_1(p_1,p_2,q_3)\right)
$$

where

$$
f_1(p_1,p_2,q_3) = \tag{99}
$$
$$
\frac{\omega(\vec{p}_1)^2}{2\omega(\vec{p}_1 - \vec{q}_3)\, r_0}\left(1 - \frac{(p_1 q_3)}{\omega(\vec{p}_1)\, r_0} - \frac{(p_1 q_3)(p_2 q_3)}{\omega(\vec{p}_1) r_0 (q_3 \overline{q}_3)}\left(2 - \frac{r_0}{\omega(\vec{p}_1)}\right)\right)
$$
$$
- \frac{\omega(\vec{p}_1)^2}{2(p_1 q_3)}\left(-\frac{\omega(\vec{q}_3)}{\omega(\vec{p}_1)} - \frac{(p_2 q_3)}{\omega(\vec{q}_3)^2} + 2\right). \tag{100}
$$

Integrating the singular terms and expanding around $\epsilon = 0$ we obtain eq. (68). Note that the D in the first term of eq. (98) arises from the γ-matrix algebra. Thus we write it as $D = 4 - 2\epsilon + c_R 2\epsilon$ to obtain the expressions in conventional dimensional regularization ($c_R = 0$) and in dimensional reduction ($c_R = 1$).

Let us now turn to the evaluation of $a_{18}^{\{2,0\}}$ needed in Section 4.5.4. We start with the expression eq. (71) and proceed in the same way as for $a_{15}^{\{2,0\}}$. We introduce the on-shell momenta $\{\not{p}_1 - \not{q}_3\} = \not{p}_1 - \not{q}_3 + \not{r}$ and $\{\not{p}_2 + \not{q}_3\} = \not{p}_2 + \not{q}_3 - \not{r}'$, where $r' = (r'_0, \vec{0})$ with $r'_0 = \rho(\vec{q}_3, \vec{p}_2) = \omega(\vec{q}_3) + \omega(\vec{p}_2) - \omega(\vec{p}_2 + \vec{q}_3)$. In order to proceed we subtract the soft singularity in eq. (71) and add it back to produce an integrand that results in a non-singular term. In the soft limit the D-dimensional delta function becomes the usual $\delta^{(D)}(P - p_1 - p_2)$ which can be pulled out from the integral.

We use the same momentum parametrization as for the self-interacting case, but because of the extra Θ function the integration ranges change to

$$
0 \le z \le \frac{2}{\Delta}\frac{(2+\Delta)}{(1-y+\Delta)} \quad \text{with} \quad -1 \le y \le \frac{\Delta}{2},
$$
$$
0 \le z \le -\frac{2}{\Delta}\frac{(2-\Delta)}{(1+y-\Delta)} \quad \text{with} \quad \frac{\Delta}{2} \le y \le 1
$$

The remaining angular integral is as given in eq. (96).

Using this momentum parametrization and expanding around the soft region gives

$$
\begin{aligned}
a_{18}^{\{2,0\}} &= -T_{ik}^a T_{kj}^a \left(\frac{\mu^2}{s}\right)^\epsilon (-ie)\, g^2 \frac{1}{2(2\pi)^{3-2\epsilon}}\, (2\pi)^D \delta^{(D-1)}(\vec{P} - \vec{p}_1 - \vec{p}_2) \\
&\quad \langle p_1|\gamma^\alpha|p_2\rangle \int d\Omega_{(2-2\epsilon)}\, dy\, dz\, z^{1-2\epsilon}\, (1-y)^{-\epsilon}\, (1+y)^{-\epsilon} \\
&\quad \times \left(\frac{1}{2z^2}\delta(\sqrt{s} - \omega(\vec{p}_1) - \omega(\vec{p}_2)) + f_2(p_1, p_2, q_3)\right) \quad (101)
\end{aligned}
$$

with

$$
\begin{aligned}
f_2(p_1, p_2, q_3) &= \frac{\omega(\vec{p}_1)^2}{2\omega(\vec{p}_1 - \vec{q}_3)\, r_0\, \omega(\vec{p}_2 + \vec{q}_3)\, r_0'} \Bigg((p_1 p_2) + (p_1 q_3) - (p_2 q_3) \\
&+ \omega(\vec{p}_1) r_0 - \omega(\vec{p}_1) r_0' - \frac{r_0 r_0'}{2} + \frac{(p_1 q_3) r_0}{2\omega(\vec{p}_1)} + \frac{(p_2 q_3) r_0'}{2\omega(\vec{p}_1)} \\
&- \frac{(p_1 q_3)(p_2 q_3)}{2\omega(\vec{p}_1)^2} - \frac{1}{2(q_3 \overline{q}_3)}\left(\frac{(p_1 q_3)^3}{\omega(\vec{p}_1)^2}\left(1 + \frac{r_0}{2\omega(\vec{p}_1)}\right)\right. \\
&- \frac{(p_2 q_3)^3}{\omega(\vec{p}_1)^2}\left(1 - \frac{r_0'}{2\omega(\vec{p}_1)}\right) + \left(2 + \frac{r_0}{\omega(\vec{p}_1)} - \frac{r_0'}{\omega(\vec{p}_1)} - \frac{r_0 r_0'}{\omega(\vec{p}_1)^2}\right) \\
&\left.\left((p_1 q_3)^2 + (p_2 q_3)^2\right)\right) \Bigg)\delta(\sqrt{s} - \omega(\vec{p}_1) - \omega(\vec{p}_2) - r_0 + r_0') \\
&- \frac{\omega(\vec{p}_1)^2}{2\omega(\vec{q}_3)^2}\delta(\sqrt{s} - \omega(\vec{p}_1) - \omega(\vec{p}_2)). \quad (102)
\end{aligned}
$$

Upon performing the integration of the singular terms explicitly and expanding in ϵ we get eq. (73). In this case the expression is the same in conventional dimensional regularization and dimensional reduction.

Finally we turn to the evaluation of $a_1^{\{1,1\}}$ needed in Section 4.5.5, proceeding as in the previous cases. We subtract the soft and collinear singular parts and integrate them analytically. In both limits the D-dimensional delta function takes its usual form $\delta^{(D)}(P - p_1 - p_2)$. Thus, the delta function is independent of the integration variables and can be taken outside the integral, as in the one-gluon exchange terms.

We can use the same momentum parametrization and integration regions as the self-interacting case as we have the same Θ function in both cases.

Taking the $z \to 0$ and $y \to 1$ limits of the above terms we obtain

$$
\begin{aligned}
a_1^{\{1,1\}} &= (-ie)\, g^2\, T_{ik}^a T_{kj}^a \left(\frac{\mu^2}{s}\right)^\epsilon \frac{1}{2(2\pi)^{3-2\epsilon}} (2\pi)^D \delta^{(D-1)}(\vec{P} - \vec{p}_1 - \vec{p}_2) \\
&\quad \langle p_1 | \gamma^\alpha | p_2 \rangle \int d\Omega_{(2-2\epsilon)}\, dy\, dz\, z^{1-2\epsilon} (1-y)^{-\epsilon} (1+y)^{-\epsilon} \qquad (103) \\
&\quad \times \left(\frac{4 - 4z + (D-2)z^2}{2(1-y)z^2} \delta(\sqrt{s} - \omega(\vec{p}_1) - \omega(\vec{p}_2)) + f_3(p_1, p_2, q_3) \right)
\end{aligned}
$$

where

$$
\begin{aligned}
f_3(p_1, p_2, q_3) &= \frac{\omega(\vec{p}_1)^2}{2\omega(\vec{p}_1 - \vec{q}_3) r_0(\{p_1 - q_3\} q_3)} \Bigg(r_0^2 + 2\, r_0\, \omega(\vec{p}_1) \\
&\quad - (p_1 q_3)\left(2 + \frac{r_0}{\omega(\vec{p}_1)}\right) + \frac{(p_1 q_3)(p_2 q_3)}{(q_3 \overline{q}_3)}\left(2 + \frac{2 r_0}{\omega(\vec{p}_1)} + \frac{r_0^2}{2\omega(\vec{p}_1)^2}\right) \Bigg) \\
&\quad + \frac{\omega(\vec{p}_1)^2}{2\omega(\vec{p}_1 - \vec{q}_3) r_0 (p_2 q_3)} \Bigg(2(p_1 p_2) + (p_1 q_3)\left(2 + \frac{r_0}{\omega(\vec{p}_1)} - \frac{(p_2 q_3)}{\omega(\vec{p}_1)^2}\right) \\
&\quad - 2(p_2 q_3) + 2 r_0 \omega(\vec{p}_1) - \frac{(p_1 q_3)^2}{(q_3 \overline{q}_3)}\left(2 + \frac{r_0}{\omega(\vec{p}_1)}\right. \\
&\quad \left. + \frac{(p_1 q_3)}{\omega(\vec{p}_1)^2}\left(1 + \frac{r_0}{2\omega(\vec{p}_1)}\right)\right) \Bigg) \delta(\sqrt{s} - \omega(\vec{p}_1) - \omega(\vec{p}_2) - r_0) \\
&\quad - \omega(\vec{p}_1)^2 \Bigg(\frac{(p_1 p_2)}{(p_1 q_3)(p_2 q_3)} + \frac{\omega(\vec{q}_3)}{\omega(\vec{p}_1)(p_1 q_3)} + \frac{(p_2 q_3)}{2(p_1 q_3)\omega(\vec{q}_3)^2} \\
&\quad - \frac{(p_1 q_3)}{2(p_2 q_3)\omega(\vec{q}_3)^2} - \frac{2}{(p_1 q_3)} \Bigg) \delta(\sqrt{s} - \omega(\vec{p}_1) - \omega(\vec{p}_2)). \qquad (104)
\end{aligned}
$$

Integrating the singular terms with $D = 4 - 2\epsilon + c_R 2\epsilon$ and expanding around $\epsilon = 0$ we obtain eq. (76).

Acknowledgments

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